Digital Signal Processing

The Real Discrete Fourier Transform

Today's Topics

- The Real Discrete Fourier Transform
  - Decomposition
  - Synthesis
- Discussion of the different types of signals and Fourier Transforms
- The Real DFT - Specifics
  - Samples
  - Basis functions
  - Synthesis equation
  - Scaling

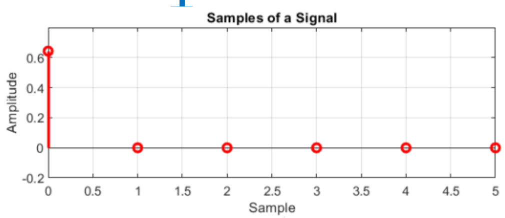
Decomposition of Signals

- In discussion of linear systems we discussed decomposing a signal into various parts
    - Impulse Decomposition – Breaking into impulses
        - Step Decomposition
    - Even and Odd Function Decomposition
    - Interlaced Decomposition
    - Fourier Decomposition -- Our focus for today

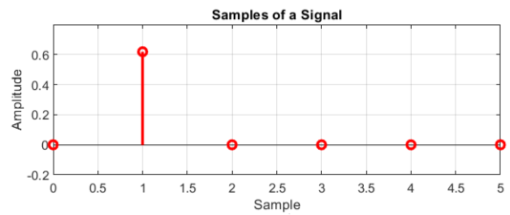
Impulse Decomposition -- Review

- What if we decompose the signal into impulses at each sample

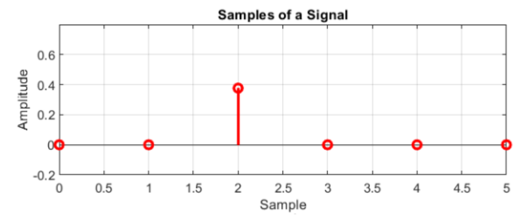
1



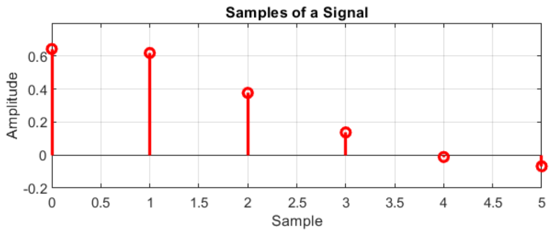
2



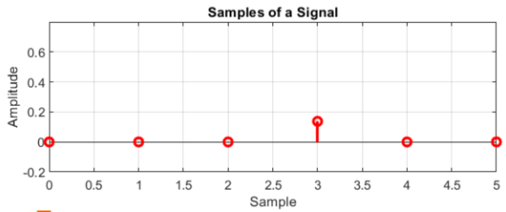
3



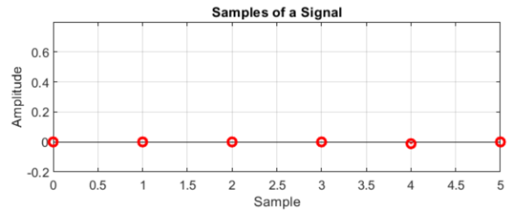
Full Signal



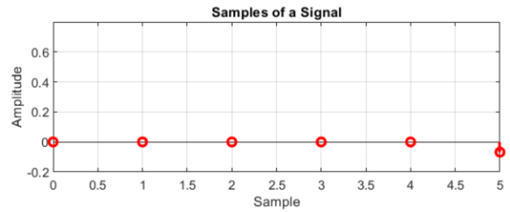
4



5



6



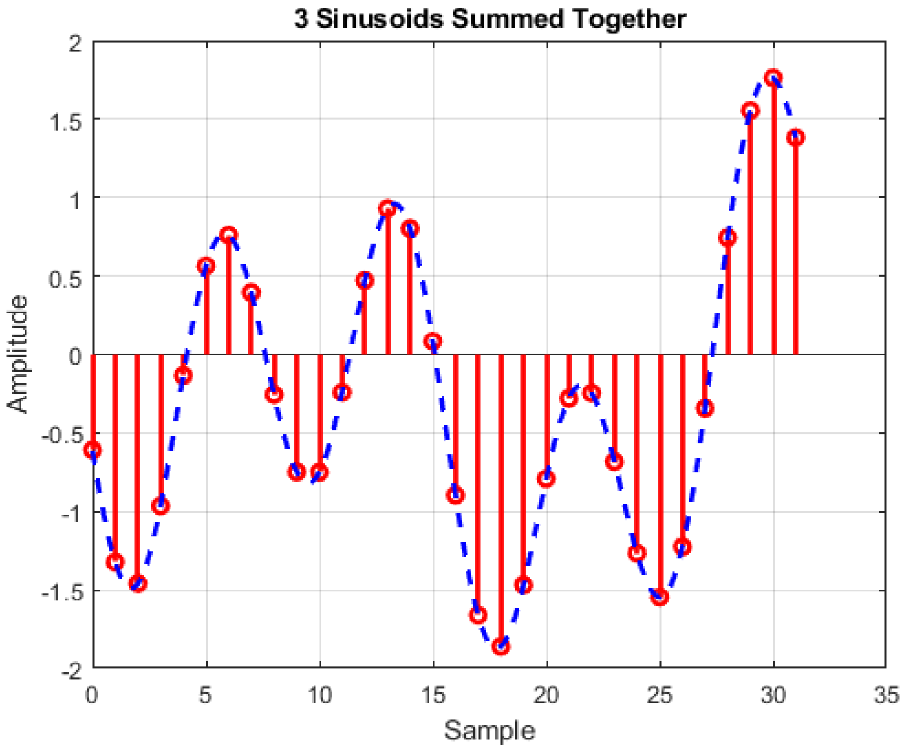
Impulse Decomposition Review

- We used impulse decomposition extensively in convolution

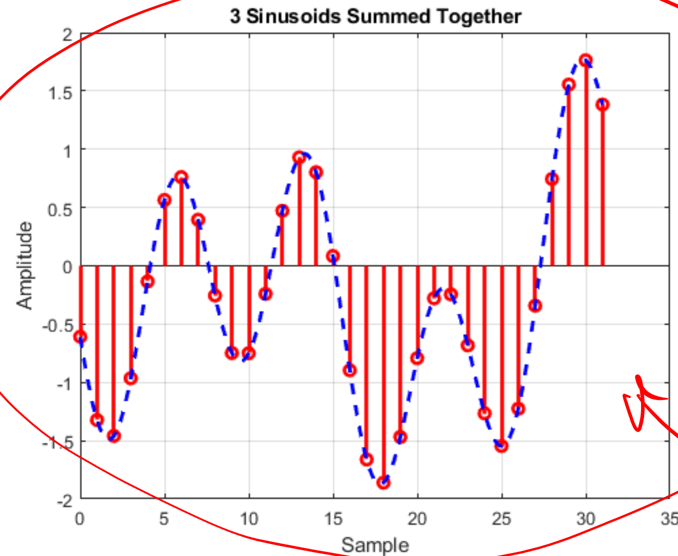
- Decomposed the input signal and applied the system impulse response, then combined

Fourier Decomposition

- Decompose the signal into a set of COSINE and SINE waves at different frequencies



3 Sinusoids added together



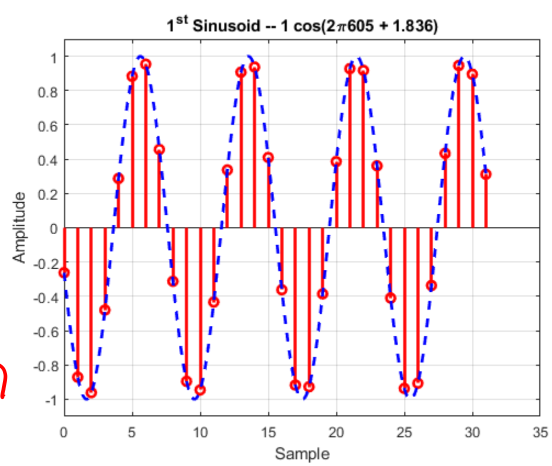
3 Sinusoids added together

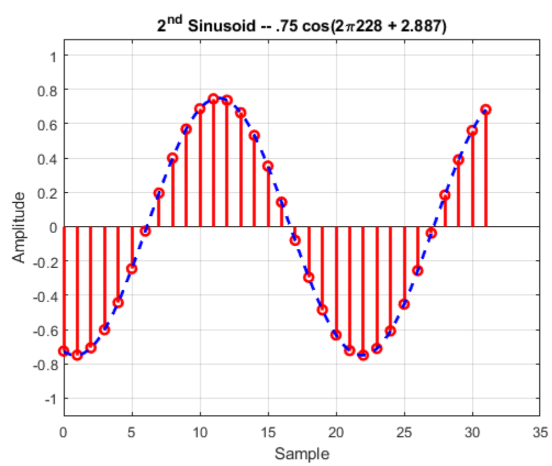
$1 \sin(2\pi 605 t + \phi)$ 1.836

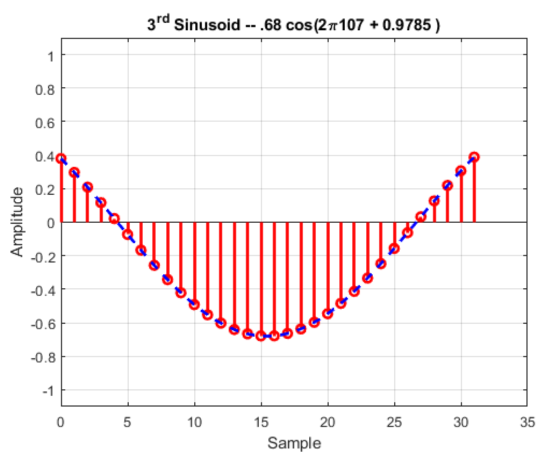
$605Hz, \phi = 1.836\ M = 1$

$228Hz, \phi = 2.89\ M = .75$

$107Hz, \phi = 0.98\ M = .68$





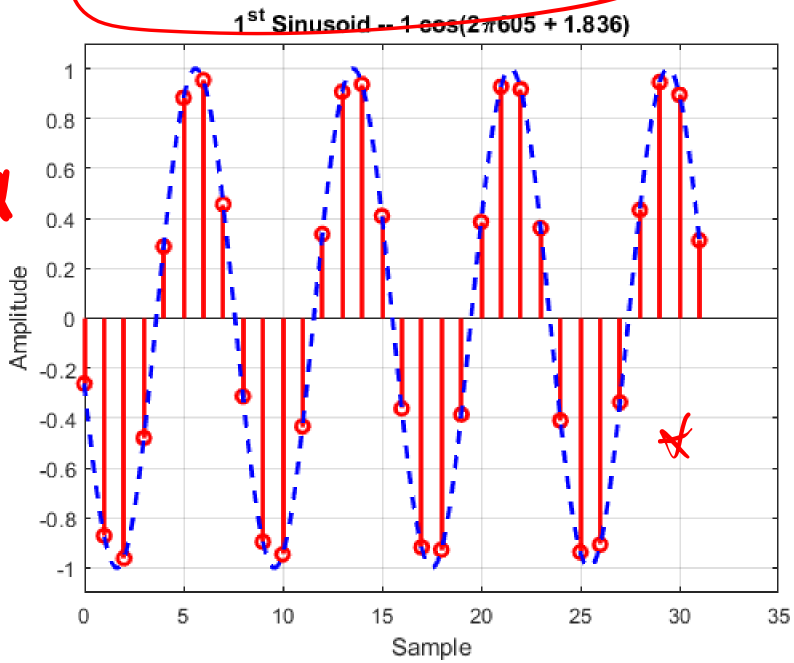


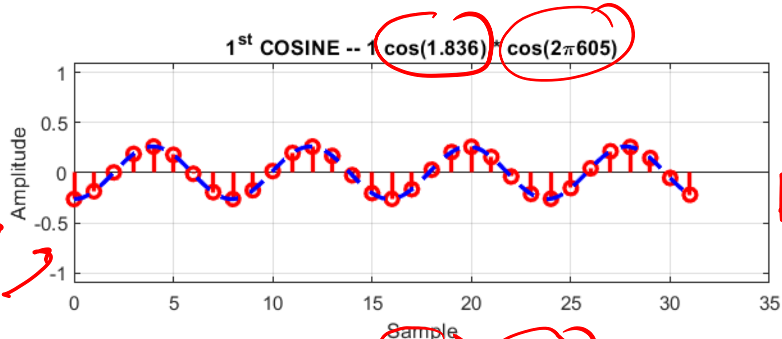
Decompose Each Sinusoid First Sinusoid

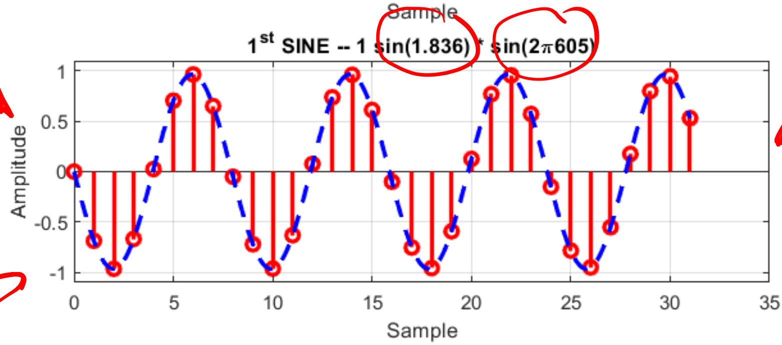
- Each sinusoid with a phase angle can be broken into a COS and SINE term

$$\cos(\omega t + \theta) = \cos(\theta)\cos(\omega t) - \sin(\theta)\sin(\omega t)$$

$605Hz, \phi = 1.836\ M = 1$



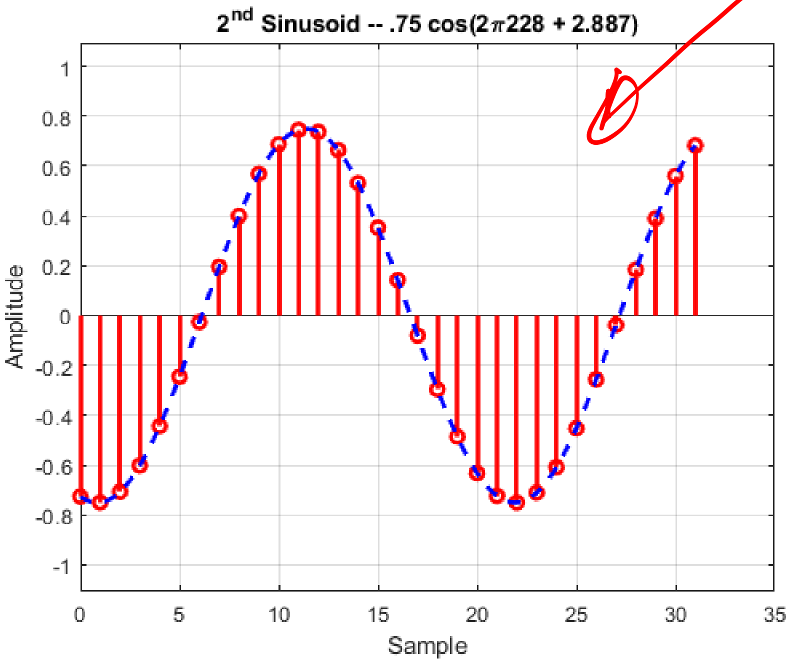


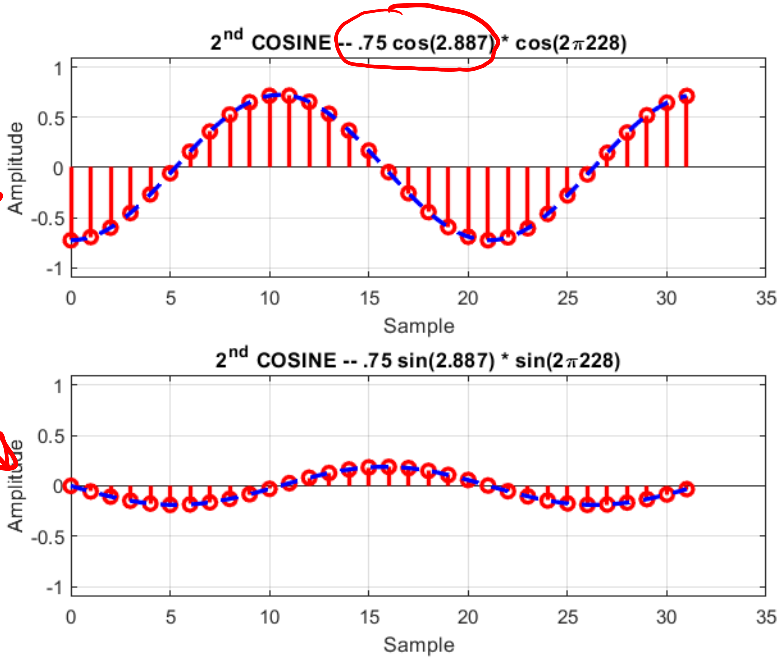


Second Sinusoid

$$cos(\omega t + \theta) = \sin(\theta)\cos(\omega t) + \cos(\theta)\sin(\omega t)$$

$228Hz, \phi = 2.89\ M = .75$

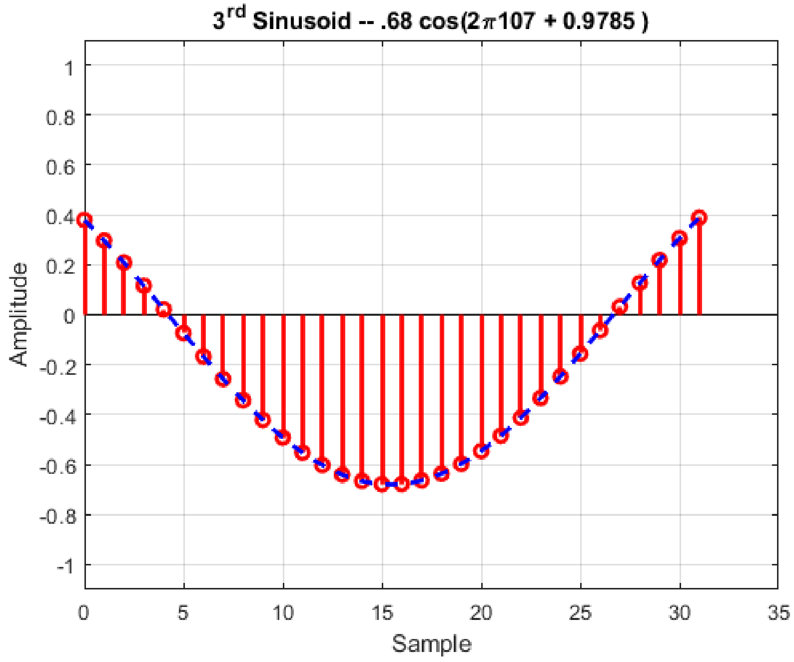


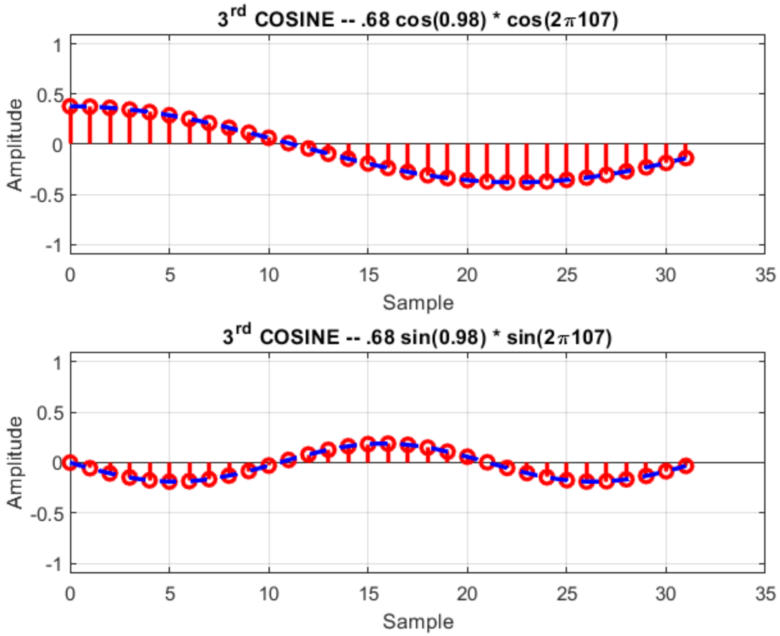


Third Sinusoid

$$cos(\omega t + \theta) = \sin(\theta)\cos(\omega t) + \cos(\theta)\sin(\omega t)$$

$107Hz, \phi = 0.98\ M = .68$

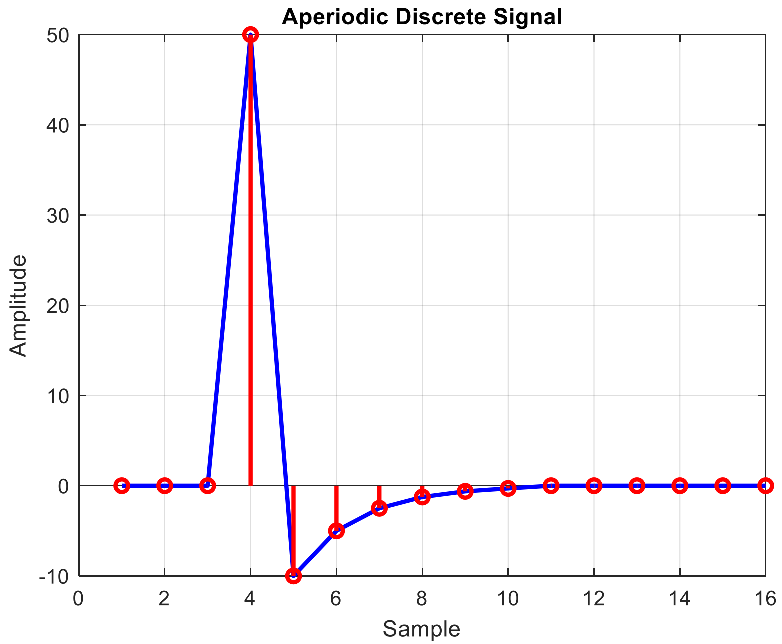




More Complex Signals

- We can easily see that a signal made up of sinusoids can be decomposed into SINE and COSINE terms

- Can I decompose this signal into COSINE and SINE signals?



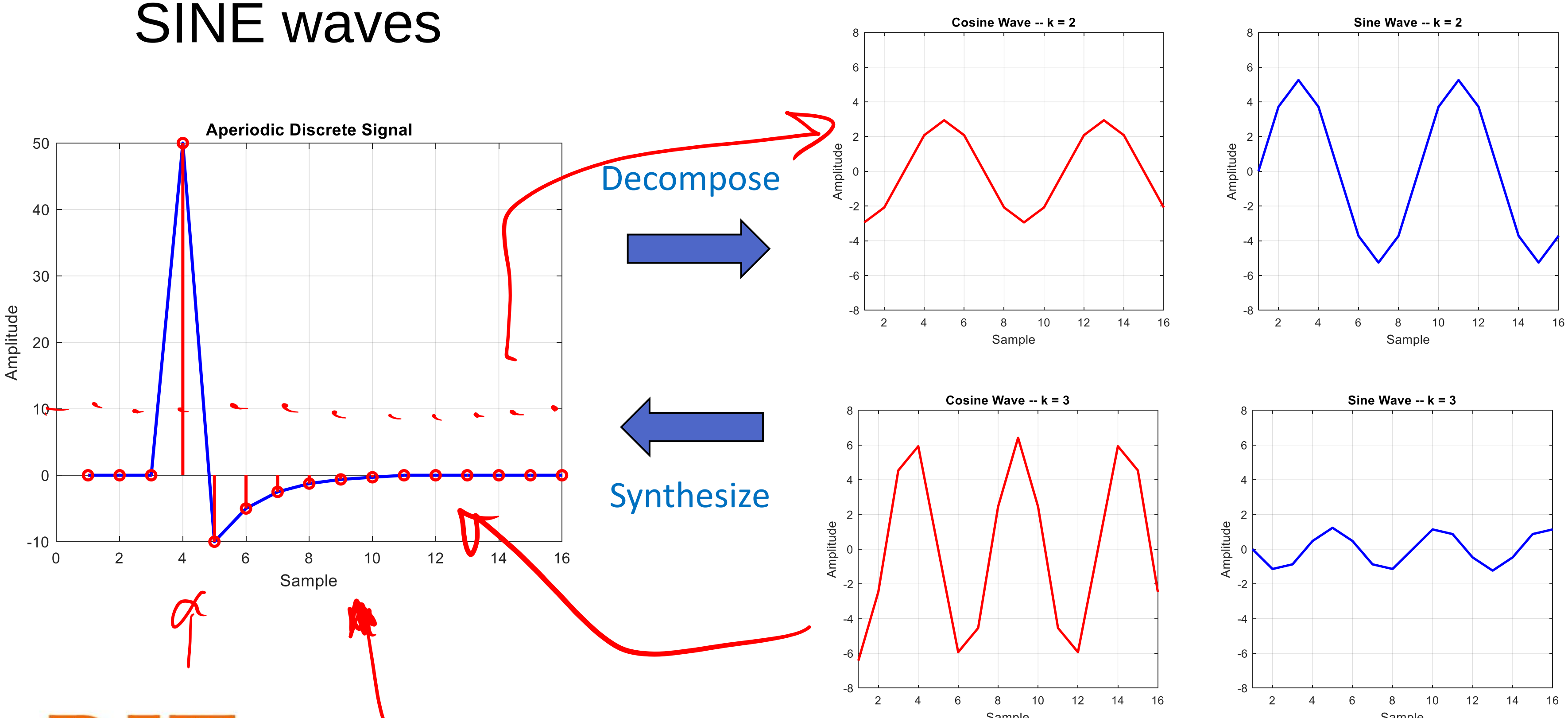
The Fourier Transform

- The Fourier Transform decomposes a signal into a set of sinusoidal signals.
- The Real Fourier Transform uses real numbers, as opposed to complex numbers
  - The complex sinusoids are broken down into the COS and SINE components

$z = r + j\,i$

Decomposition and Synthesis

- Decompose – Break a signal into COS and SINE waves
- Synthesis – Reconstruct the signal from the COS and SINE waves



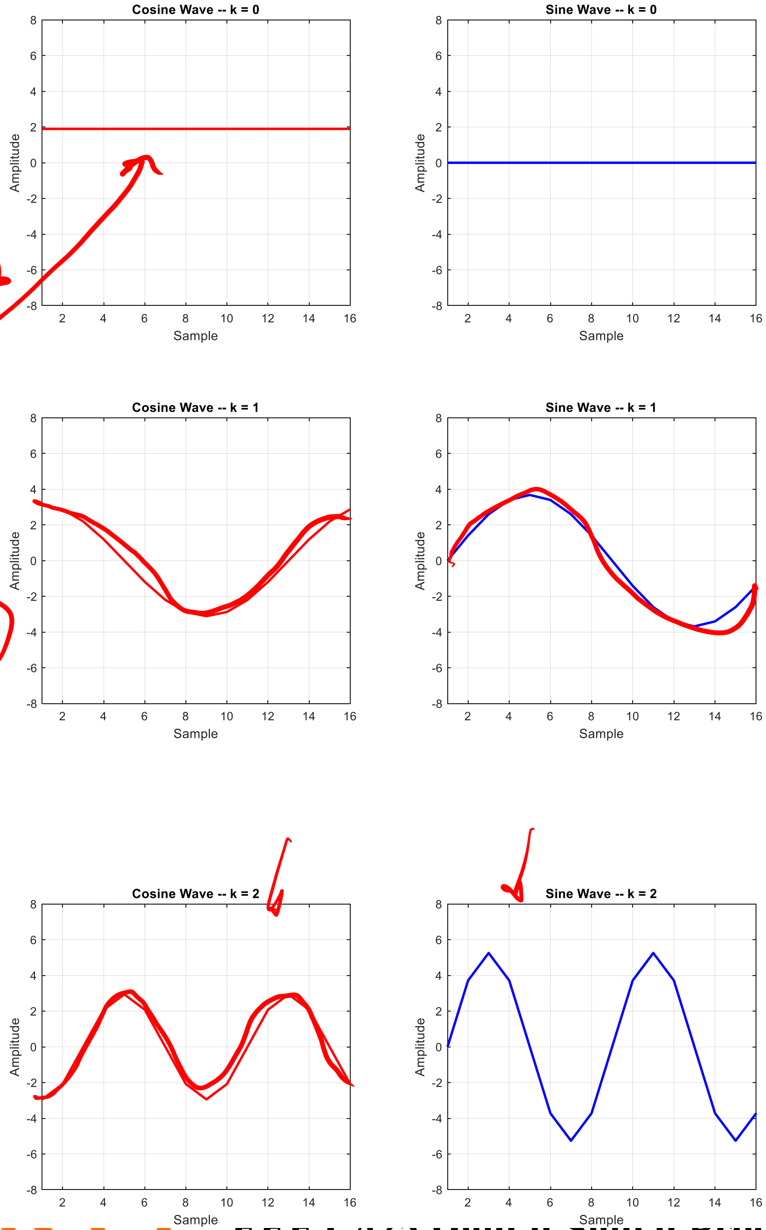
13

COS SINE COS SINE

k=0

k=1

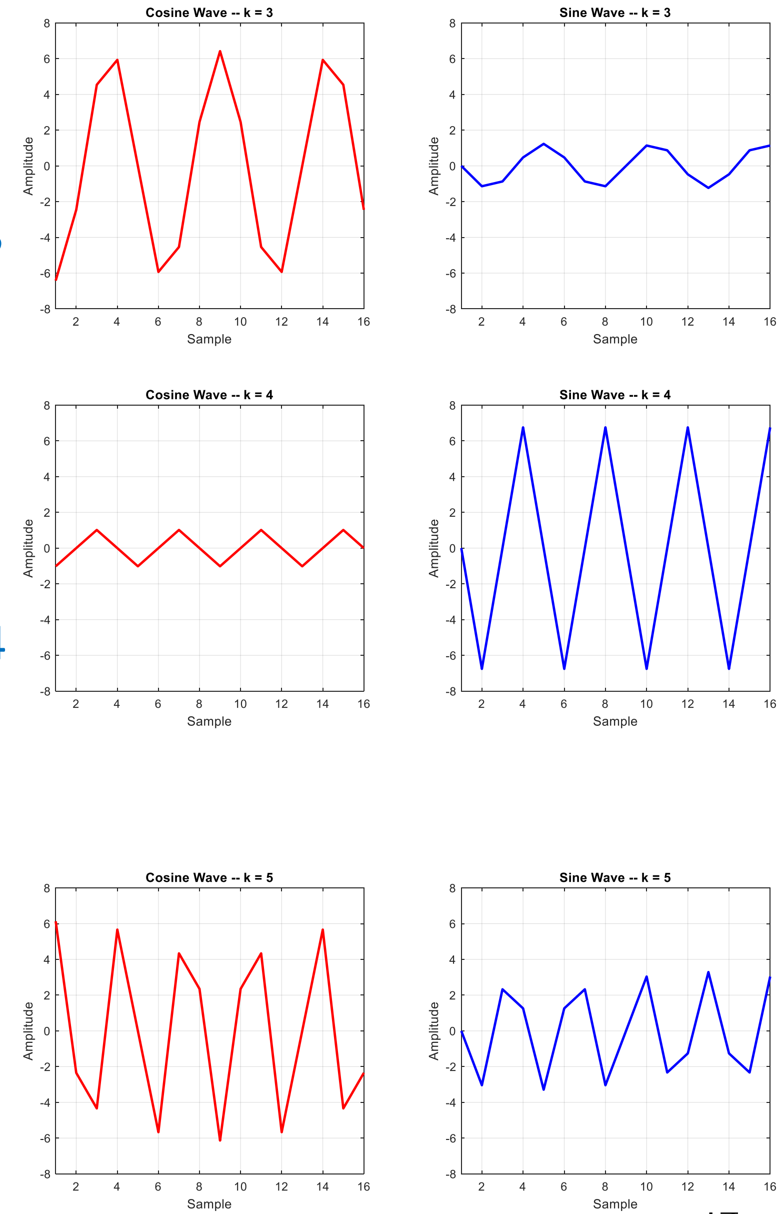
k=2



k=3

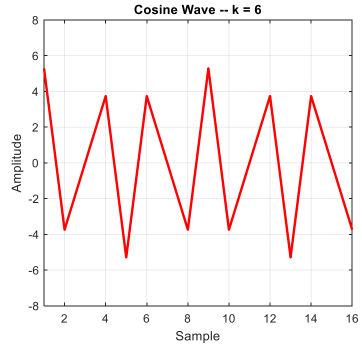
k=4

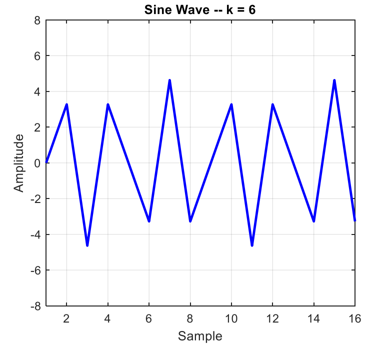
k=5



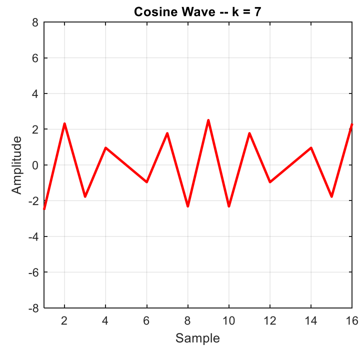
COS SINE

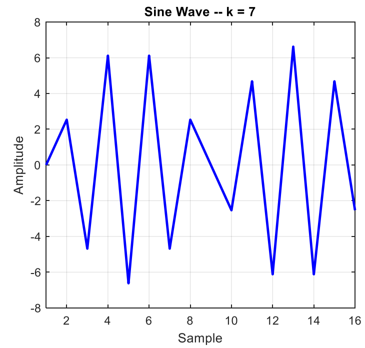
k=6



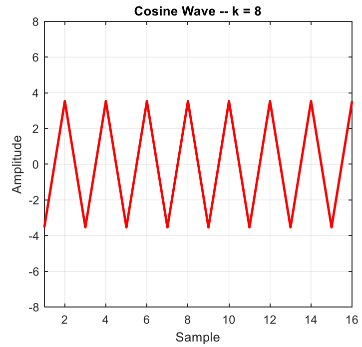


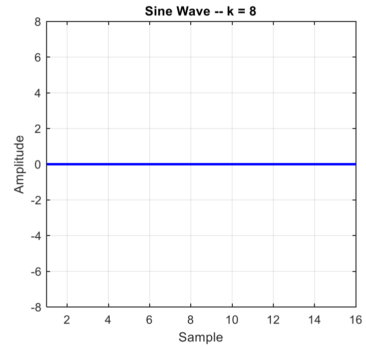
k=7





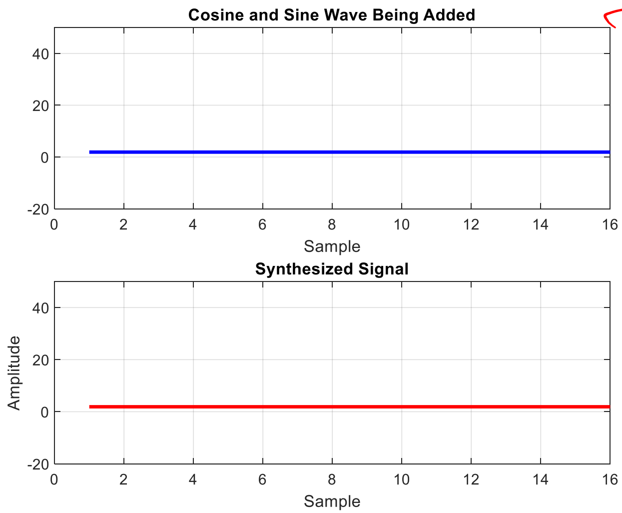
k=8



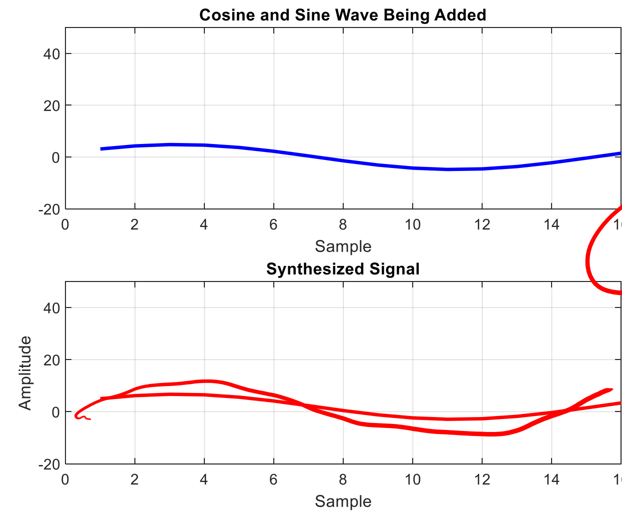


MATLAB Demo
DFT_Demo.m

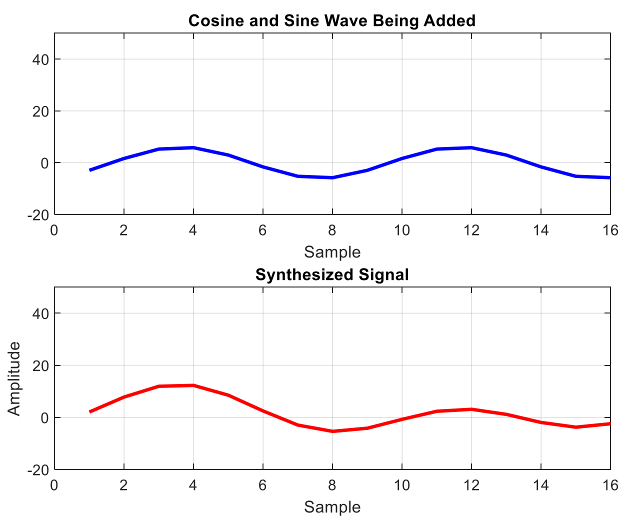
Can We Synthesize the Signal from the COS and SINE's?



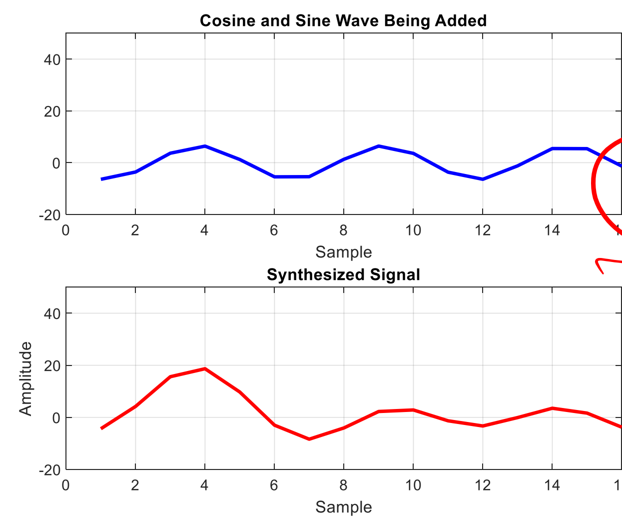
k = 0



k = 1



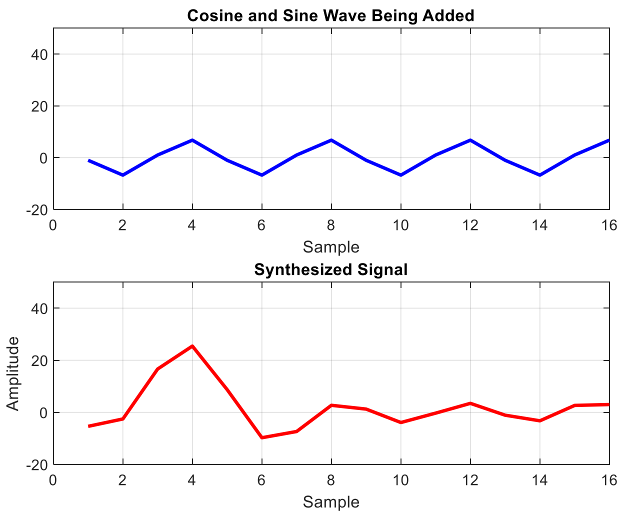
k = 2

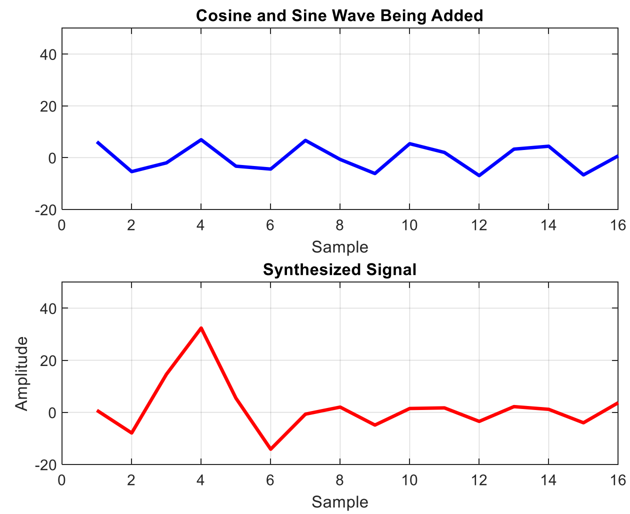


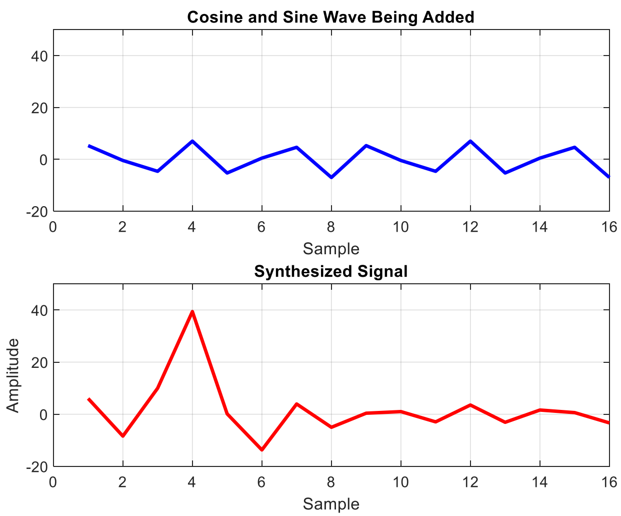
k = 3

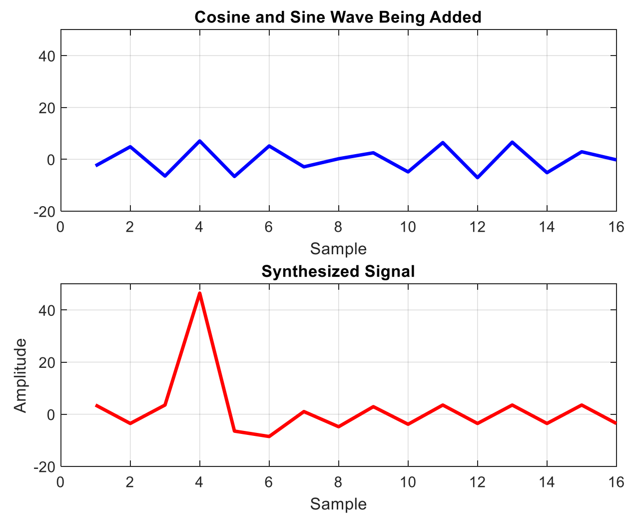
Processing

Can We Synthesize the Signal from the COS and SINE's?





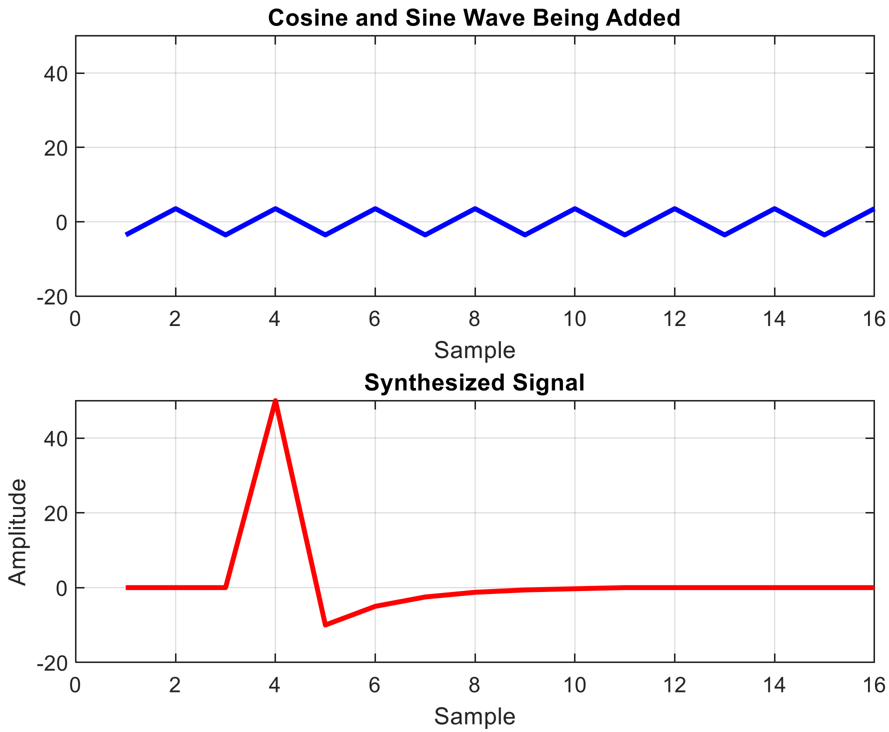




l Processing

The Original Signal Synthesized from Each COS and SINE

- The signal has been reproduced <u>exactly</u> from the 9 COS/SINE signal pairs

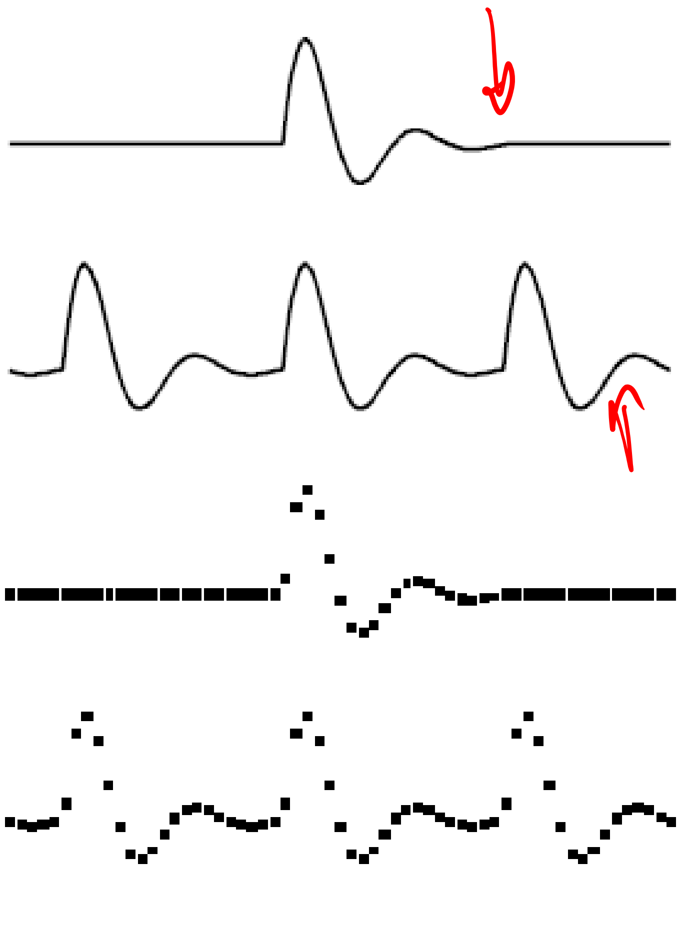


How do we decompose a signal?

- A *transform* is used to decompose the signal and break it down into the COS and SINE components

- Which *transform* is used depends on the type of signal

Characterizing Signals



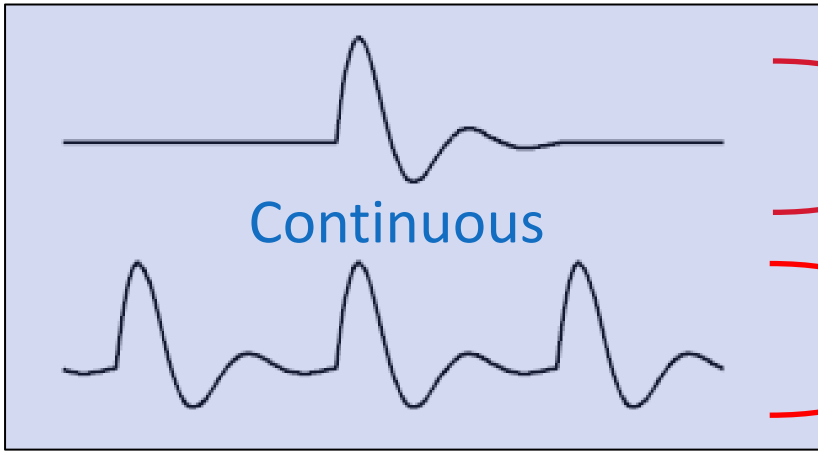
Continuous

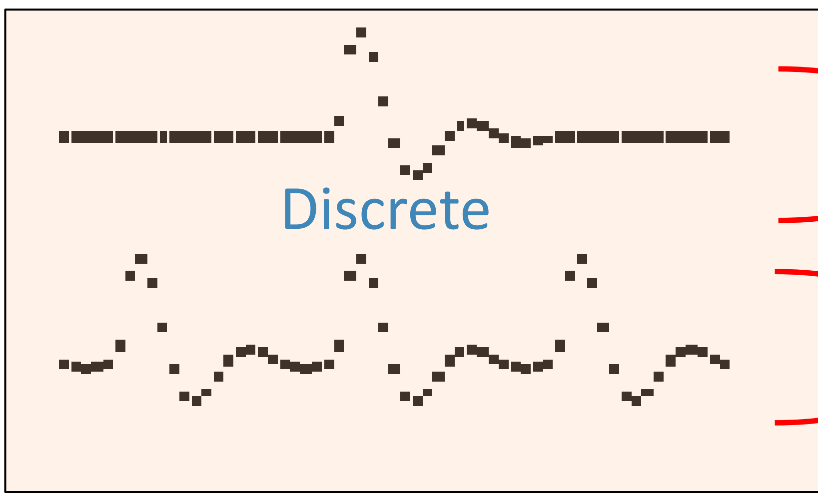
$$\int_{-\infty}^{\infty} s(t)\, e^{-j\omega t}\, dt$$

Discrete

$$\Sigma$$

Characterizing Signals





Signal Type		Transform Type
Continuous Aperiodic	→	Fourier Transform
Continuous Periodic	→	Fourier Series
Discrete Aperiodic	→	Discrete Time Fourier Transform
Discrete Periodic	→	Discrete Fourier Transform

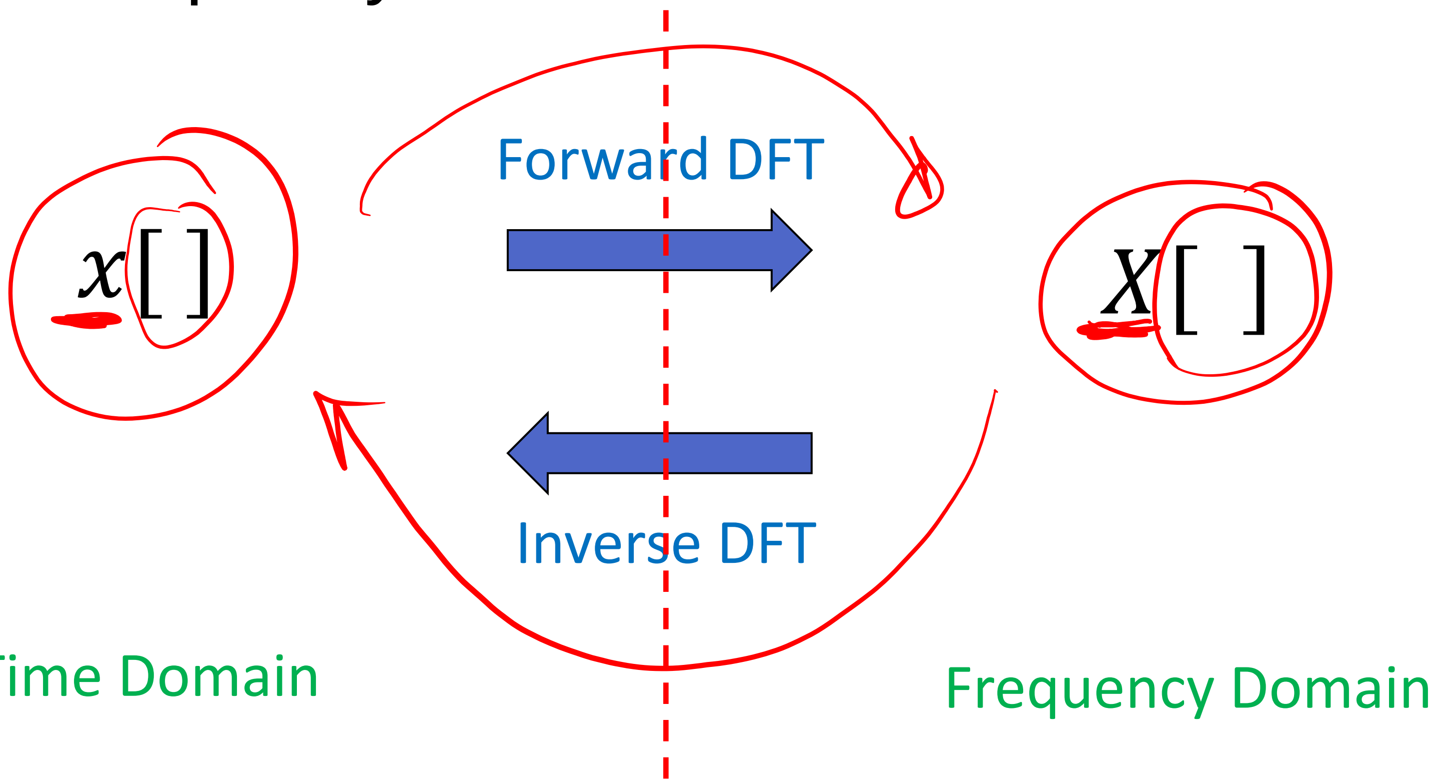
Discrete Fourier Transform

- We will be only be using the Discrete Fourier Transform (DFT)

- We will always be talking about <u>discrete</u> time samples of a signal that is assumed to be <u>periodic</u>.

Discrete Fourier Transform

- The DFT transforms a time domain signal into the frequency domain



Discrete Fourier Transform

- N samples in the time domain produce $N/2+1$ samples in the frequency domain

16

9

k=0 → 8

Forward DFT

$x[\,]$ $X[\,]$

Inverse DFT

N-Samples in the Time Domain

N/2 +1 Samples In the Frequency Domain

Time Domain

Frequency Domain

Discrete Fourier Transform

- The real part of the frequency domain signal are the COSINE amplitudes. Imaginary part are the SINE amplitudes

Frequency Domain

$$X[\ ]$$

N/2 +1 Samples In the Frequency Domain

$$Re[X]$$

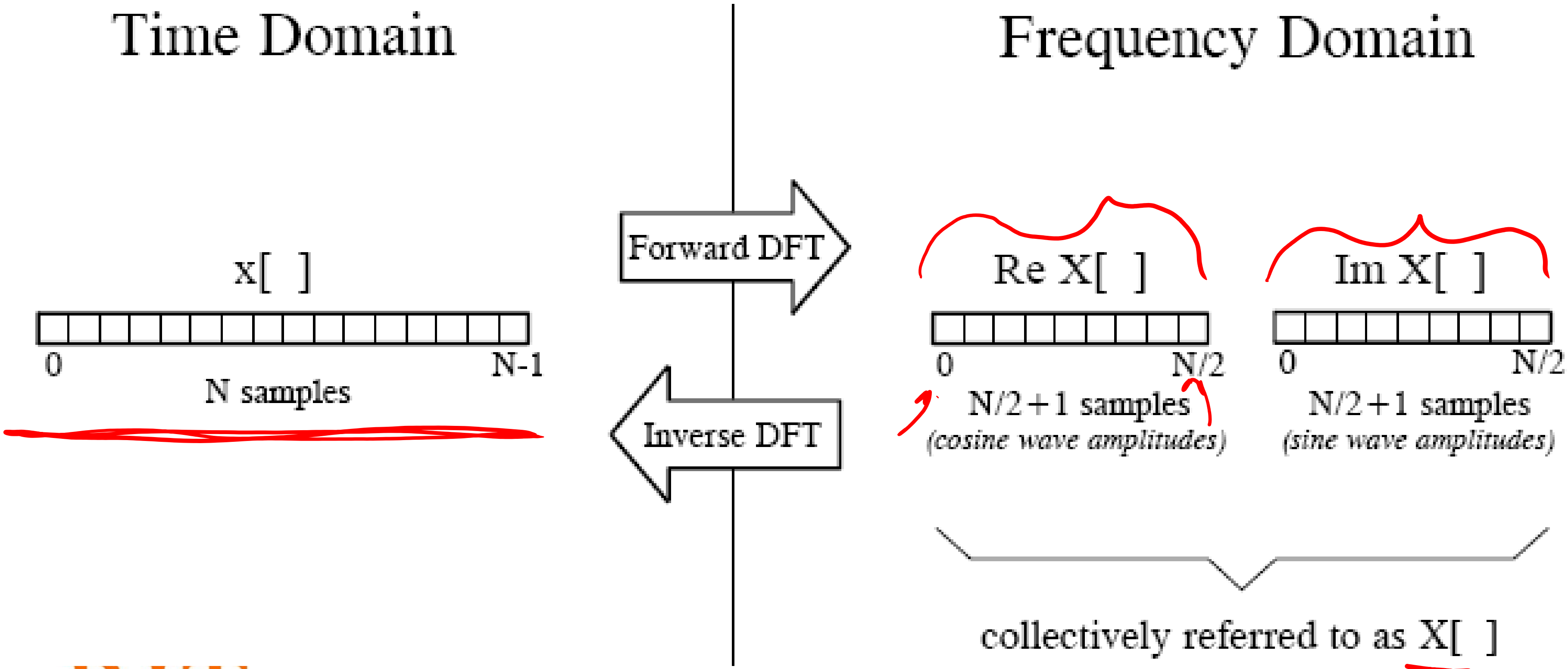
Amplitude of the COSINE waves

$$Im[X]$$

Amplitudes of the SINE waves

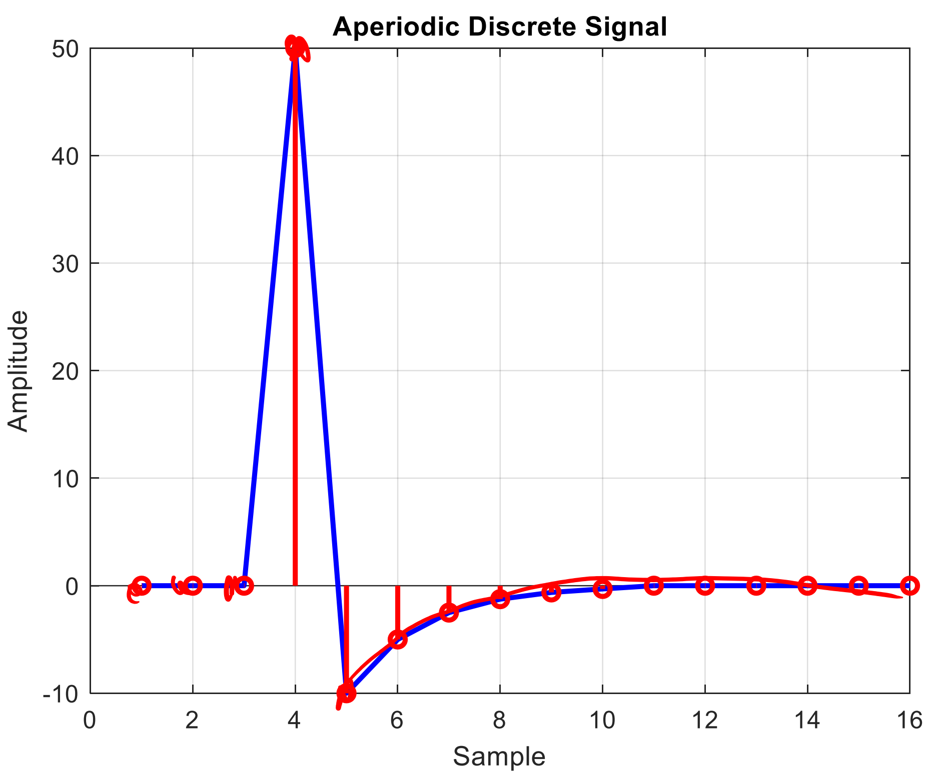
Real DFT: Time to Frequency Domain Transform

- Frequency Domain refers to the amplitude of cosines/sines



DFT of Our Previous Example

- The signal has N=16 samples in the time domain



n	x[n]
0	0
1	0
2	0
3	50
4	-10
5	-5
6	-2.5
7	-1.25
8	-0.625
9	-0.3
10	0
11	0
12	0
13	0
14	0
15	0

N/2+1

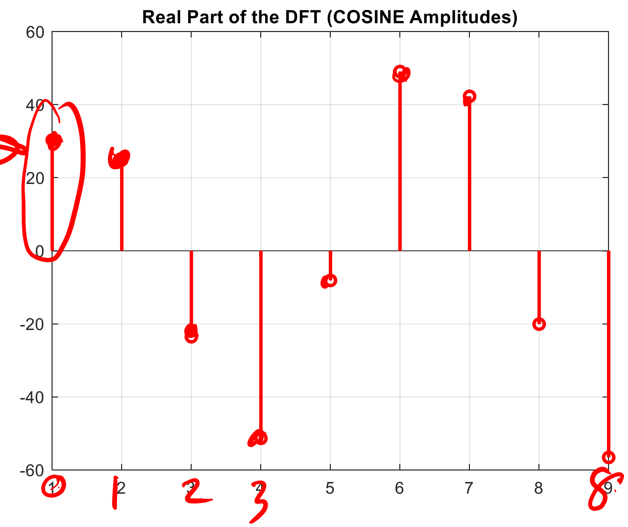
Re

X

Forward DFT Results

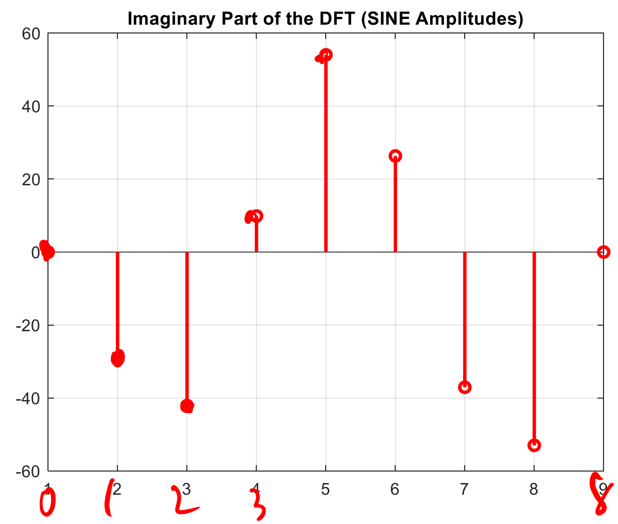
N/2+1 Samples
9 Samples
COSINE Amplitudes

n	Re[X]
0	30.33
1	24.87
2	-23.54
3	-51.36
4	-8.13
5	49.08
6	42.29
7	-20.09
8	-56.58



N/2+1 Samples
9 Samples
SINE Amplitudes

n	Im[X]
0	0.00
1	-29.44
2	-42.06
3	9.87
4	54.05
5	26.33
6	-37.06
7	-52.98
8	0.00



Forward DFT Results

N/2+1 Samples
9 Samples
COSINE Amplitudes

n	Re[X]
0	30.33
1	24.87
2	-23.54
3	-51.36
4	-8.13
5	49.08
6	42.29
7	-20.09
8	-56.58

k=0
Scaled amplitude of the ~~1st~~ COSINE

k=3
Scaled amplitude of the ~~4th~~ COSINE

N/2+1 Samples
9 Samples
SINE Amplitudes

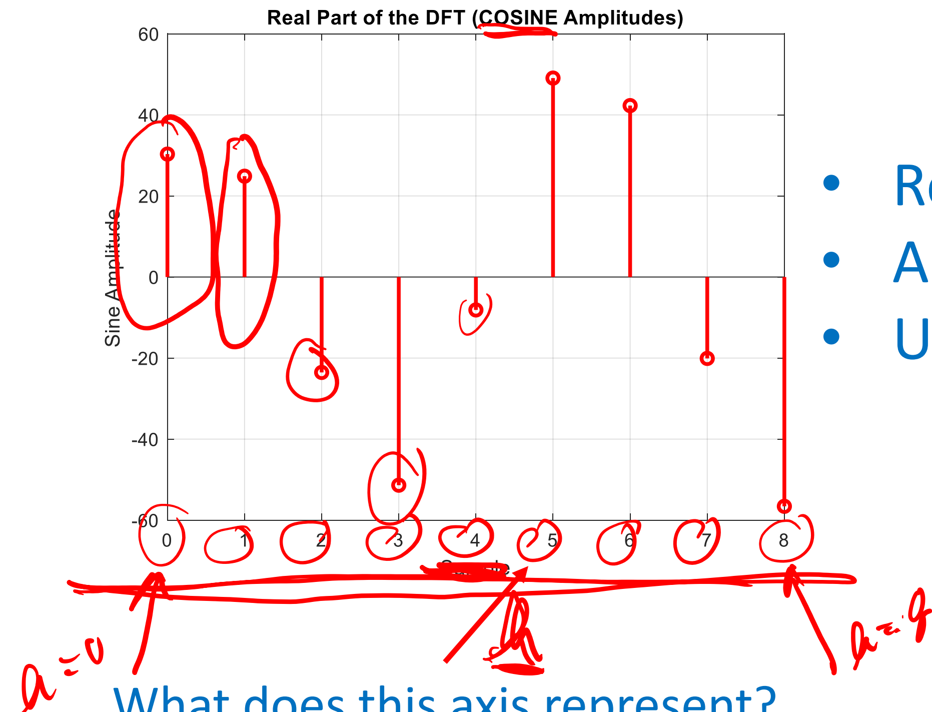
n	Im[X]
0	0.00
1	-29.44
2	-42.06
3	9.87
4	54.05
5	26.33
6	-37.06
7	-52.98
8	0.00

k=2
Scaled amplitude of the ~~3rd~~ SINE

k=6
Scaled amplitude of the ~~7th~~ SINE

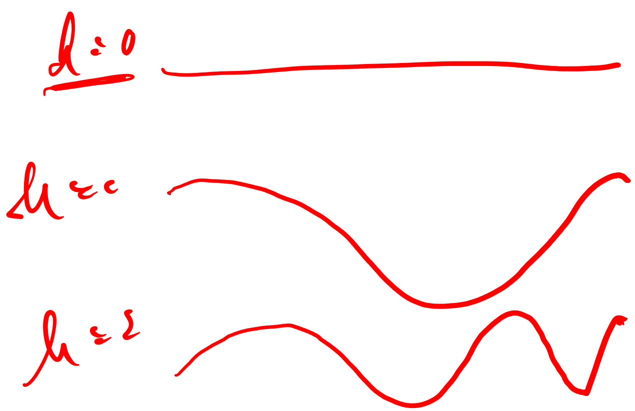
Frequency Domain Independent Variable

- What is the independent variable in the frequency domain? 4 different representations



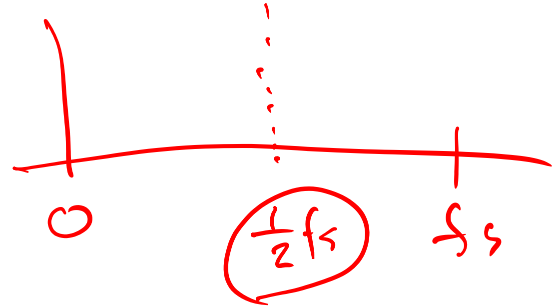
- Represents the 0 to N/2 samples
- An integer value
- Useful in programming (e.g. indexing)

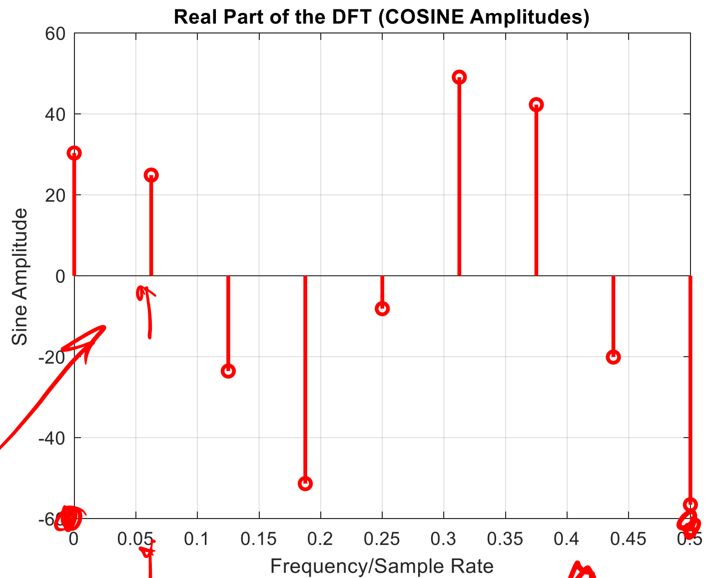
What does this axis represent?
Sample Number



Frequency Domain Independent Variable

- Fraction of the sample rate





0 $\frac{1}{2}$

- Represents fraction of the sample rate
- Maximum of 0.5 – Nyquist Rate

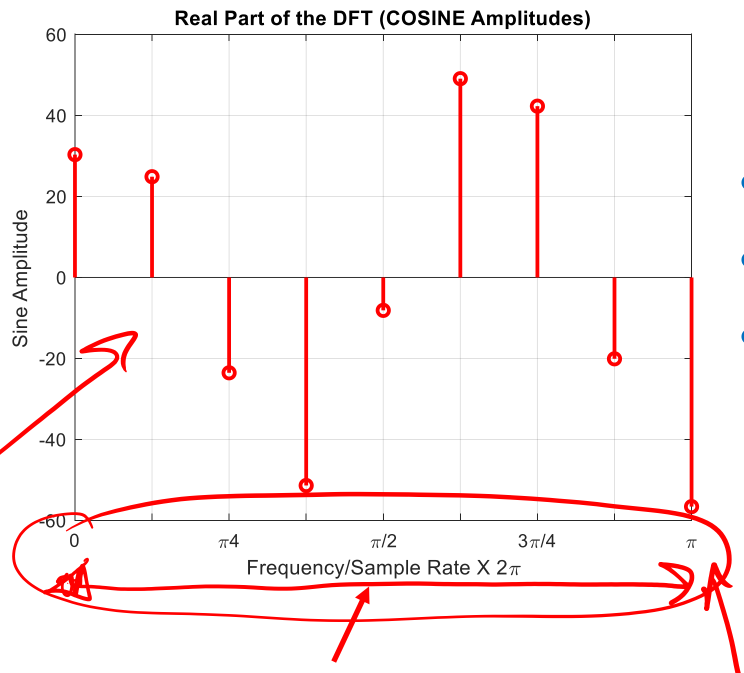
What does this axis represent?
Fraction of Sampling Rate

Frequency Domain Independent Variable

- Natural frequency in rad/sec

$f_s = 2\pi$

$\frac{f_s}{2} = \pi$



- Natural Frequency -- Radians
- Fraction of the sample rate times 2π
- Maximum of π – Nyquist Rate

What does this axis represent?

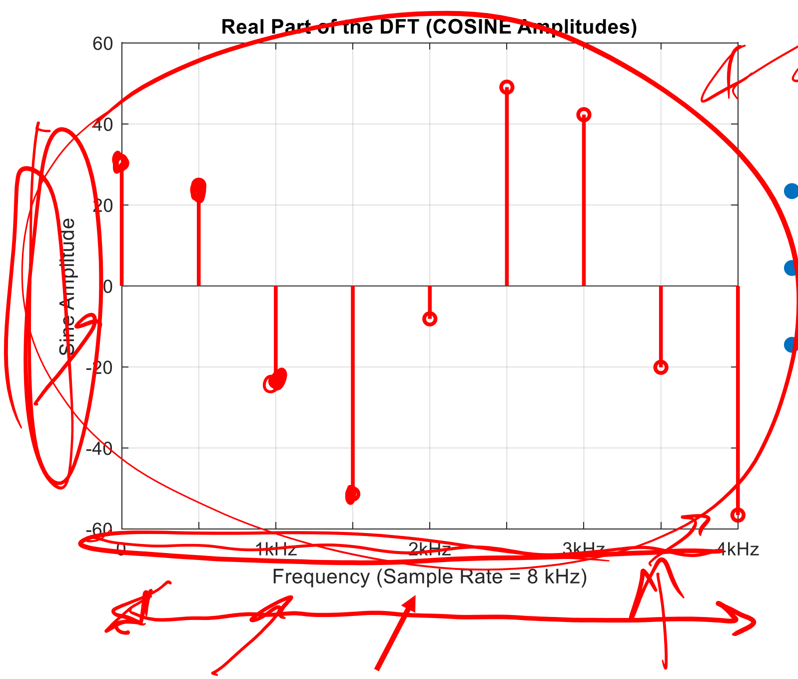
Fraction of Sampling Rate

Frequency Domain Independent Variable

- The absolute frequency

$f_s = 8\,kHz$

$\frac{f_s}{2} = 4\,kHz$



- Absolute Frequency
- Maximum of the Nyquist Rate
- Assume 8 kHz sample rate

What does this axis represent?

Fraction of Sampling Rate

What do the COS and SINE waves Represent

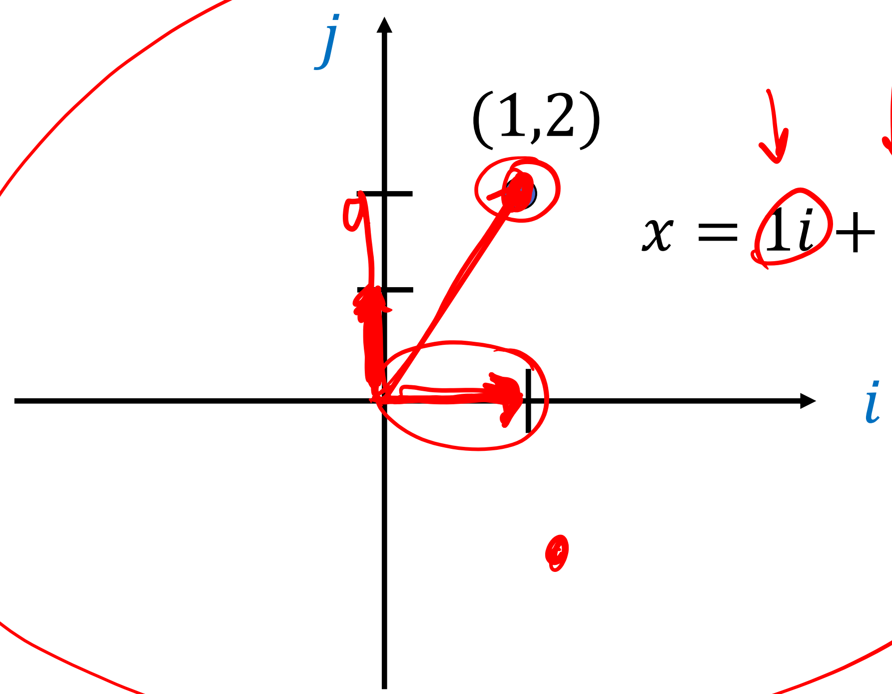
- The COSINE and SINE wave signals are BASIS functions

- What is a BASIS function?

  - A set of orthonormal functions that when linearly combined can create any function in the space
  - Orthonormal – Orthogonal and Unit Length functions

BASIS function example

- Consider the cartesian plane – Any point in the plane can be described by a linear combination of the BASIS functions $i = \binom{1}{0}$ $j = \binom{0}{1}$



$x = 1i + 2j$

BASIS Functions for the DFT

- The BASIS functions for the DFT are

$$\boxed{\begin{aligned} c_k[i] &= \cos(2\pi ki/N) \\ s_k[i] &= \sin(2\pi ki/N) \end{aligned}}$$

$k = 0 \to 8$

$k = 0$

- These represent COS and SINE functions that have a frequency of k/N

- The COS and SINE function will complete k cycles in N samples

BASIS Functions for the DFT

- The BASIS functions for the DFT are

$$c_k[i] = \cos(2\pi ki/N)$$

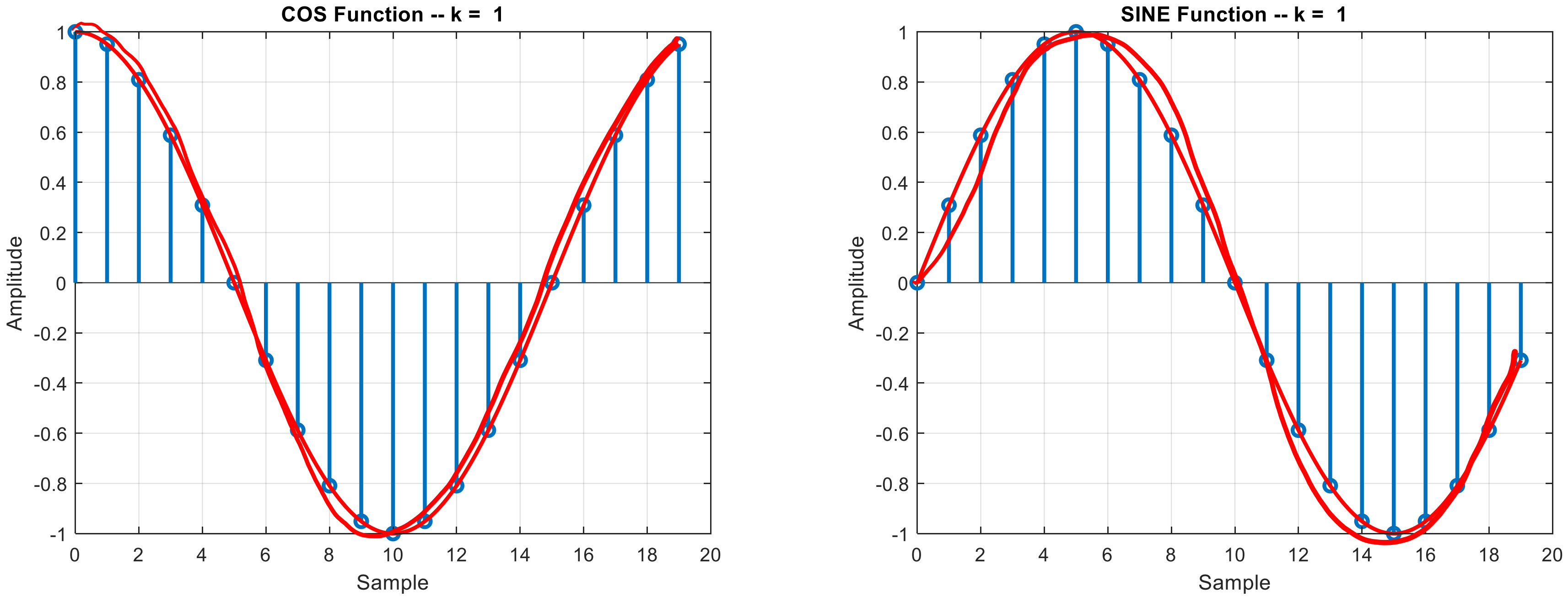
$$s_k[i] = \sin(2\pi ki/N)$$

- i goes from 0 to N-1 and represents the time domain

- k goes from 0 to N/2 and represents the frequency

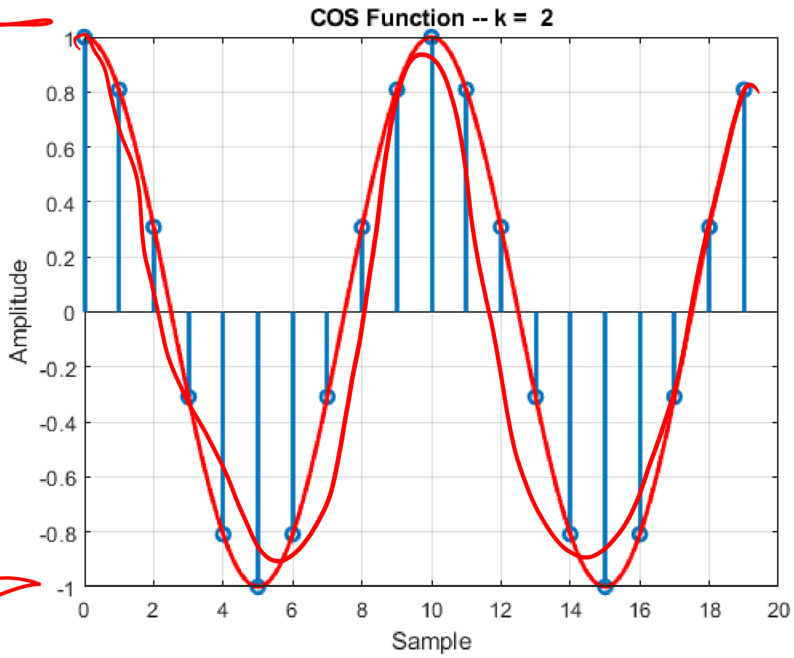
Example Basic Functions
k = 1, N = 20

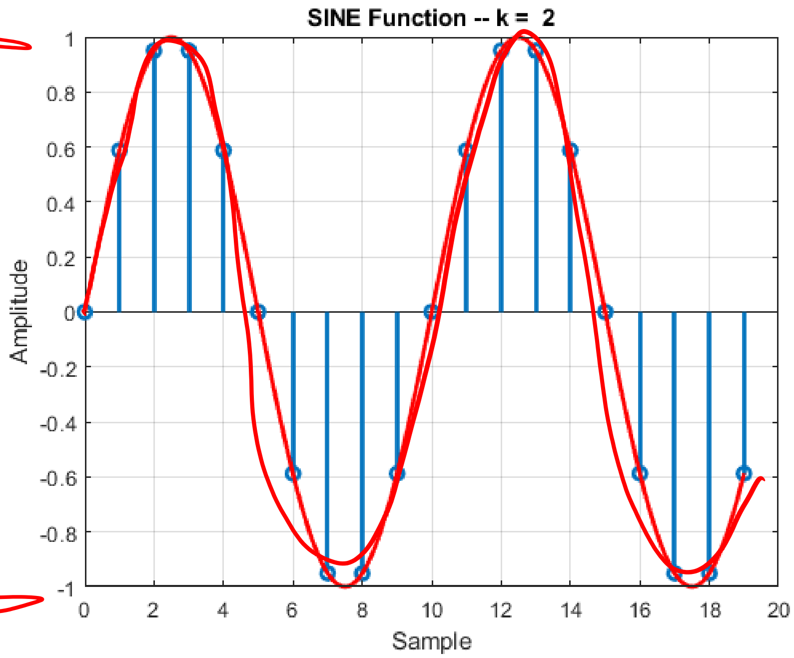
- The function completes 1 cycle in N samples



Example Basic Functions
k = 2, N = 20

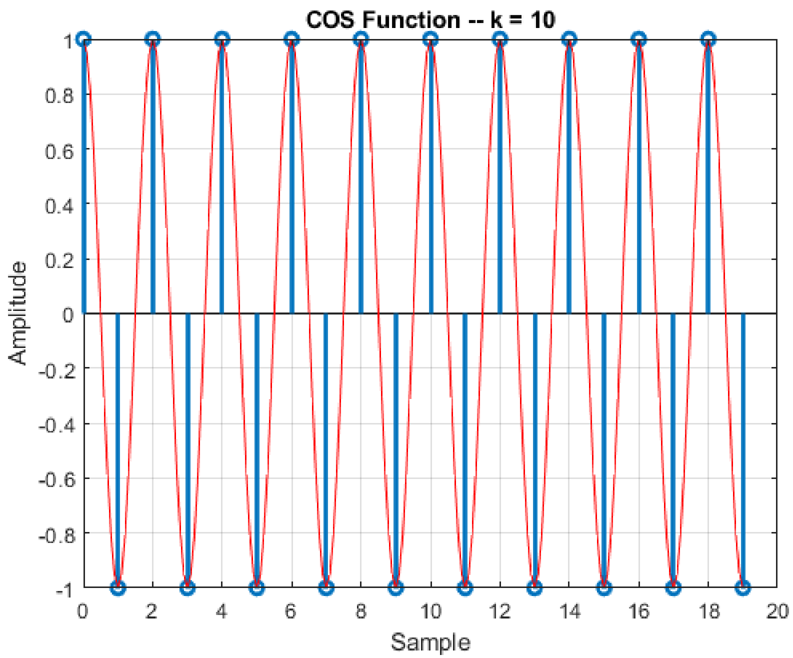
- The function completes 2 cycles in N samples

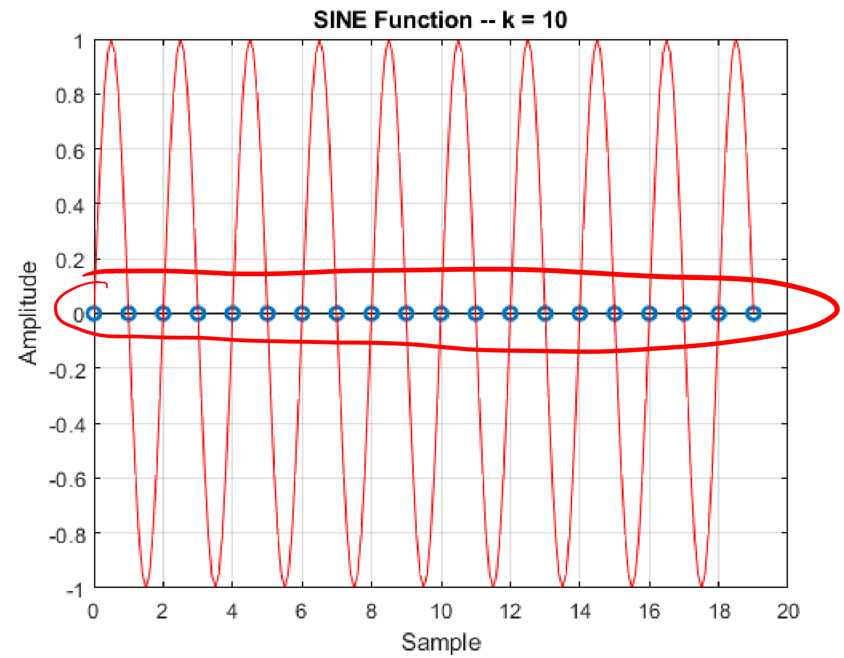




Example Basic Functions
k = N/2 = 10, N = 20

- The function completes ~~2~~ 10 cycles in N samples





But How do We Get $X[k]$?

- We *correlate* the input sequence with each COS and SINE wave at $N/2+1$ frequencies

$$Re(X[k]) = \sum_{i=0}^{N-1} x[i]\cos(2\pi k i/N)$$

$$Im(X[k]) = \sum_{i=0}^{N-1} x[i]\sin(2\pi k i/N)$$

Multiply the input sequence by N samples of the cosine and sine signals for each frequency k

Basis Functions of the DFT

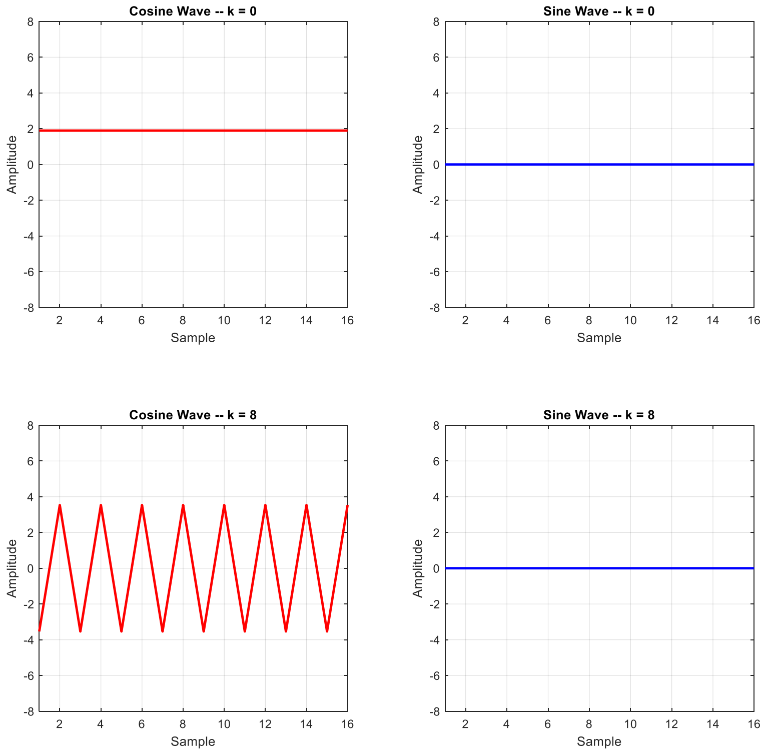
- Note that the coefficients of $Im\,X[0]$ and $Im\,X[N/2]$ are always zero.

COS SINE

$Re(X[0])$ is the DC component

k=0

k=8



Imaginary values are all zero

From previous example with N=16

Imaginary values are all zero

Computing the DFT

- Create the COS and SINE signals at k = 0
- Then multiply by each point of the input and sum

			k=0, N=16			k=0, N=16	
i	x[i]		COS(2*pi*k*i/N)	SIN(2*pi*k*i/N)		x[i] COS(2*pi*k*i/N)	x[i] SIN(2*pi*k*i/N)
0	0		1	0		0	0
1	0		1	0		0	0
2	0		1	0		0	0
3	50		1	0		50	0
4	-10		1	0		-10	0
5	-5		1	0		-5	0
6	-2.5		1	0		-2.5	0
7	-1.25		1	0		-1.25	0
8	-0.625		1	0		-0.625	0
9	-0.3		1	0		-0.3	0
10	0		1	0		0	0
11	0		1	0		0	0
12	0		1	0		0	0
13	0		1	0		0	0
14	0		1	0		0	0
15	0		1	0		0	0
					X[k]	30.325	0

RIT

Computing the DFT

- Create the COS and SINE signals at k = 1
- Then multiply by each point of the input and sum

			k=1, N=16			k=1, N=16	
i	x[i]		COS(2*pi*k*i/N)	SIN(2*pi*k*i/N)		x[i] COS(2*pi*k*i/N)	x[i] SIN(2*pi*k*i/N)
0	0		1.000	0.000		0.000	0.000
1	0		0.924	0.383		0.000	0.000
2	0		0.707	0.707		0.000	0.000
3	50		0.383	0.924		19.134	46.194
4	-10		0.000	1.000		0.000	-10.000
5	-5		-0.383	0.924		1.913	-4.619
6	-2.5		-0.707	0.707		1.768	-1.768
7	-1.25		-0.924	0.383		1.155	-0.478
8	-0.625		-1.000	0.000		0.625	0.000
9	-0.3		-0.924	-0.383		0.277	0.115
10	0		-0.707	-0.707		0.000	0.000
11	0		-0.383	-0.924		0.000	0.000
12	0		0.000	-1.000		0.000	0.000
13	0		0.383	-0.924		0.000	0.000
14	0		0.707	-0.707		0.000	0.000
15	0		0.924	-0.383		0.000	0.000
					X[k]	24.872	29.443

Computing the DFT

- Create the COS and SINE signals at k = 2
- Then multiply by each point of the input and sum

			k=2, N=16			k=2, N=16	
i	**x[i]**		**COS(2*pi*k*i/N)**	**SIN(2*pi*k*i/N)**		**x[i] COS(2*pi*k*i/N)**	**x[i] SIN(2*pi*k*i/N)**
0	0		1.000	0.000		0.000	0.000
1	0		0.707	0.707		0.000	0.000
2	0		0.000	1.000		0.000	0.000
3	50		-0.707	0.707		-35.355	35.355
4	-10		-1.000	0.000		10.000	0.000
5	-5		-0.707	-0.707		3.536	3.536
6	-2.5		0.000	-1.000		0.000	2.500
7	-1.25		0.707	-0.707		-0.884	0.884
8	-0.625		1.000	0.000		-0.625	0.000
9	-0.3		0.707	0.707		-0.212	-0.212
10	0		0.000	1.000		0.000	0.000
11	0		-0.707	0.707		0.000	0.000
12	0		-1.000	0.000		0.000	0.000
13	0		-0.707	-0.707		0.000	0.000
14	0		0.000	-1.000		0.000	0.000
15	0		0.707	-0.707		0.000	0.000
					X[k]	-23.541	42.063

BASIS Functions for the DFT

- Each signal can be represented by the linear combination of:
  - $N/2 + 1$ COSINE waves
  - $N/2 + 1$ SINE waves

Linear combination of N/2+1 terms

$$x[n] = \underbrace{\sum_{k=0}^{N/2} \underbrace{Re(\bar{X}[k])}_{\text{Cosine Magnitude}} \ \underbrace{\cos(2\pi ki/N)}_{\text{Cosine Basis Function -- k}}} + \sum_{k=0}^{N/2} \underbrace{Im(\bar{X}[k])}_{\text{Sine Magnitude}} \underbrace{\sin(2\pi ki/N)}_{\text{Sine Basis Function -- k}}$$

What is $\bar{X}$?

- $X[]$ is the values that we get when we perform the DFT on the time domain signal
- $Re(X)$ is the real portion
- $Im(X)$ is the imaginary portion

- We need to scale these values when synthesizing the original signal from the SINE and COSINE signals

Scaling $Re[X]$ and $Im[X]$

$$Re(\bar{X}[k]) = \frac{Re(X[k])}{N/2}$$

$$Im(\bar{X}[k]) = -\frac{Im(X[k])}{N/2}$$

Except for two special cases:

$$Re(\bar{X}[0]) = \frac{Re(X[0])}{N}$$ First frequency (DC)

$$Re(\bar{X}[N/2]) = \frac{Re(X[N/2])}{N}$$ Last frequency

Why the Scaling?

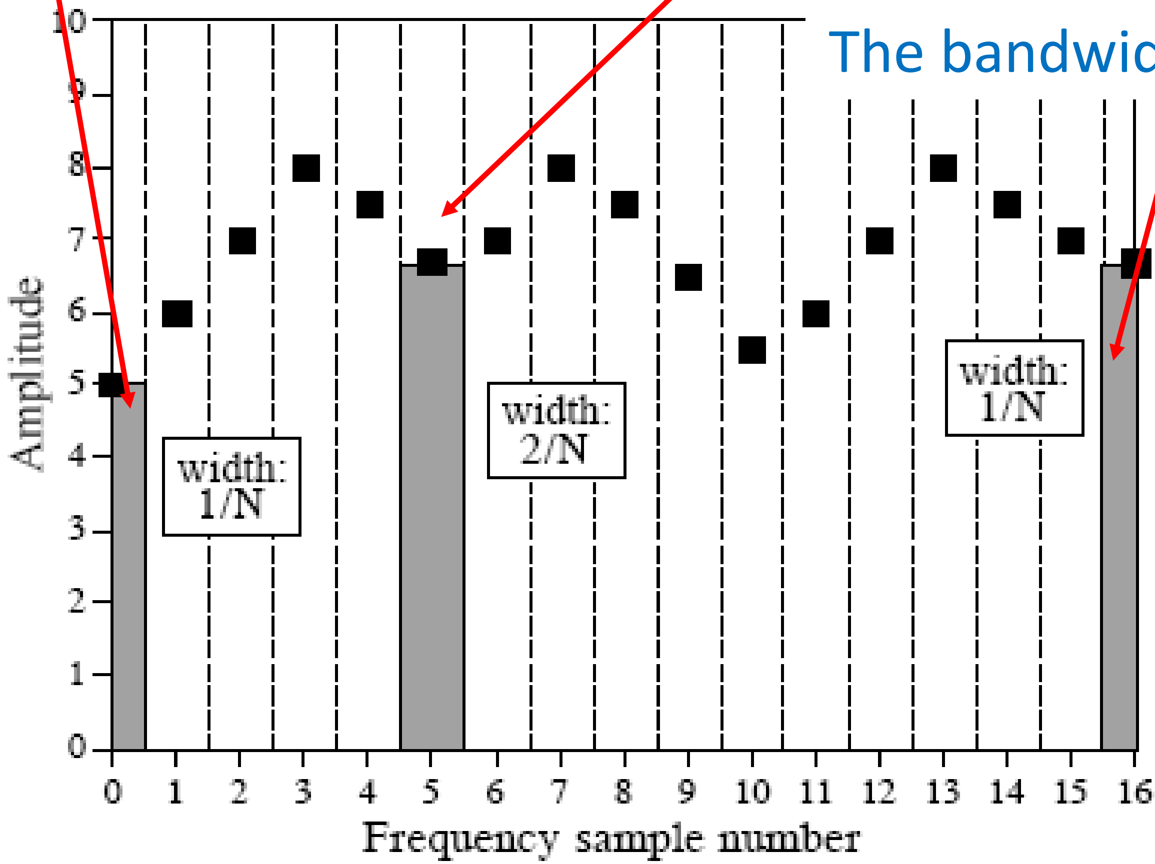
- The frequency domain coefficients are spectral densities
- Signal amplitude per unit bandwidth.
- The bandwidth is different for the end frequencies (0 and N/2)

Scaling of Coefficients

The bandwidth here is 1/N

The bandwidth here is 2/N

The bandwidth here is 1/N



Scaling $Re[X]$ and $Im[X]$

- Scale by N/2 except for $Re(X[0])$ and $Re(X[N/2])$ where the scale is N

n	Re[X]	Scale	Re[Xbar]
0	30.33	1/16	1.90
1	24.87	1/8	3.11
2	-23.54	1/8	-2.94
3	-51.36	1/8	-6.42
4	-8.13	1/8	-1.02
5	49.08	1/8	6.13
6	42.29	1/8	5.29
7	-20.09	1/8	-2.51
8	-56.58	1/16	-3.54

Special cases

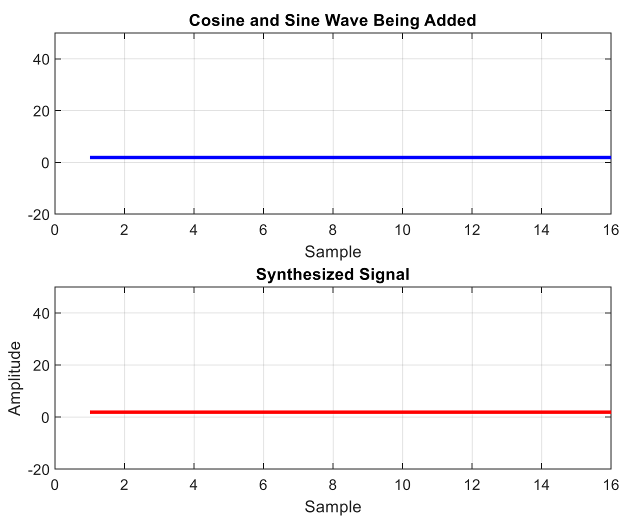
n	Im[X]	Scale	Im[Xbar]
0	0.00	1/8	0.00
1	-29.44	1/8	3.68
2	-42.06	1/8	5.26
3	9.87	1/8	-1.23
4	54.05	1/8	-6.76
5	26.33	1/8	-3.29
6	-37.06	1/8	4.63
7	-52.98	1/8	6.62
8	0.00	1/8	0.00

Repeating Our Earlier Example
Linear Combination of COS and SINE Waves

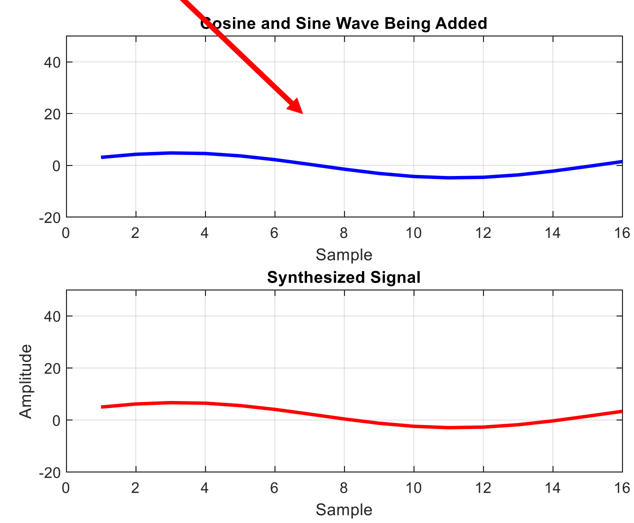
This signal is the sum of the cosine and sine at k=1 for N =16 samples

$$Re(\bar{X}[1])\cos(2\pi(1)i/N) + Im(\bar{X}[1])\,\sin(2\pi(1)/N)$$

k=0



k=1

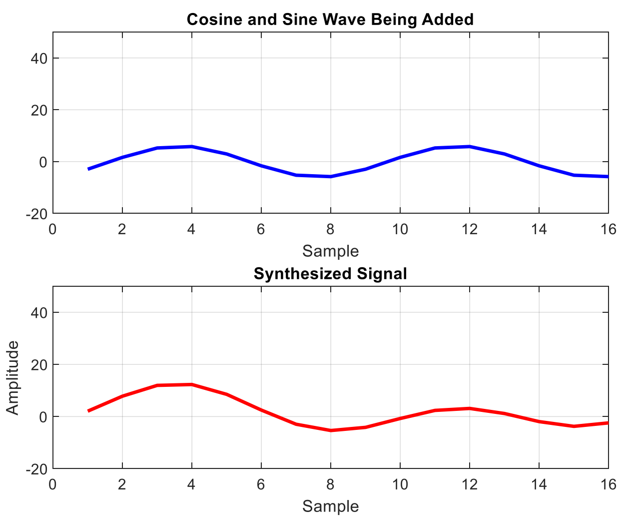


Repeating Our Earlier Example
Linear Combination of COS and SINE Waves

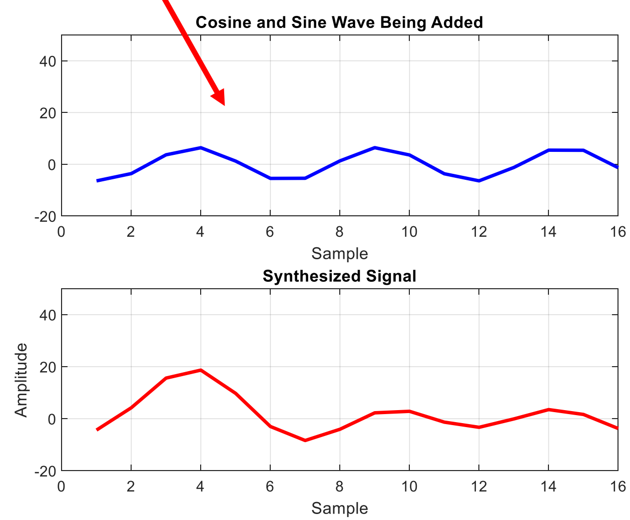
This signal is the sum of the cosine and sine at k=4 for N =16 samples

$$Re(\bar{X}[4])\cos(2\pi(4)i/N) + Im(\bar{X}[4])\ \sin(2\pi(4)/N)$$

k=3

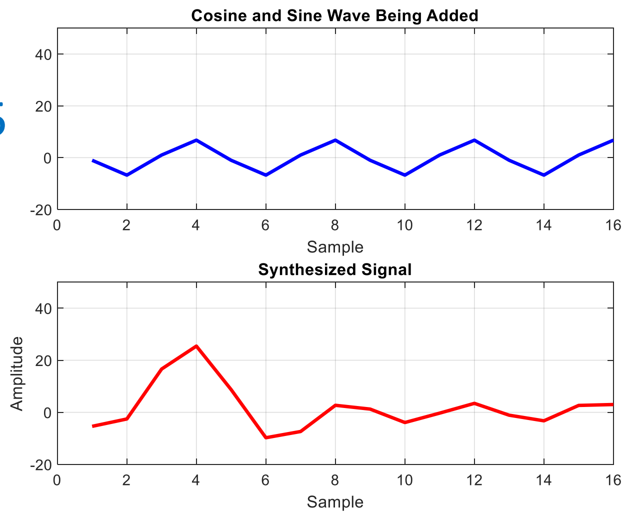


k=4

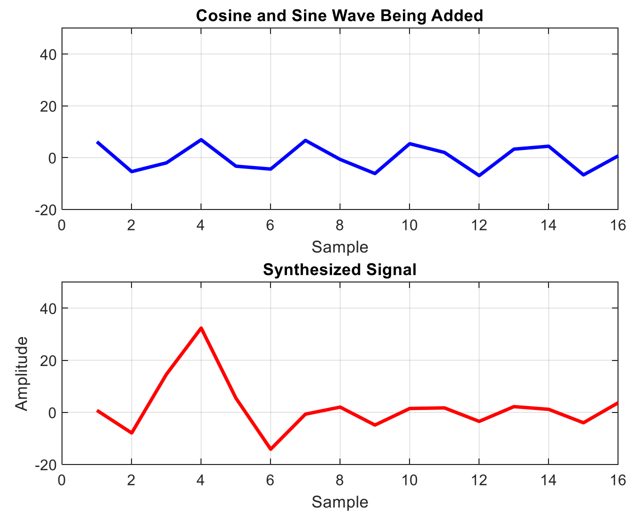


Can We Synthesize the Signal from the COS and SINE's?

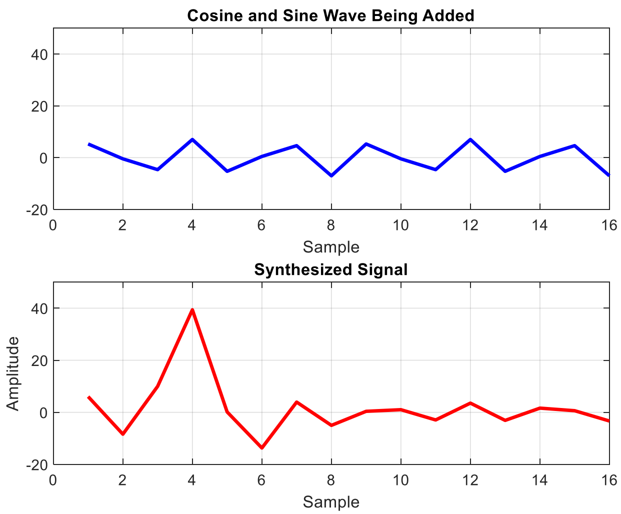
k=5



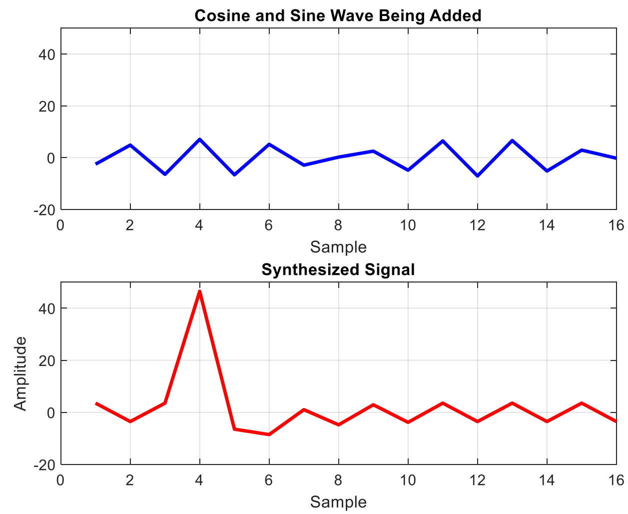
k=6



k=7



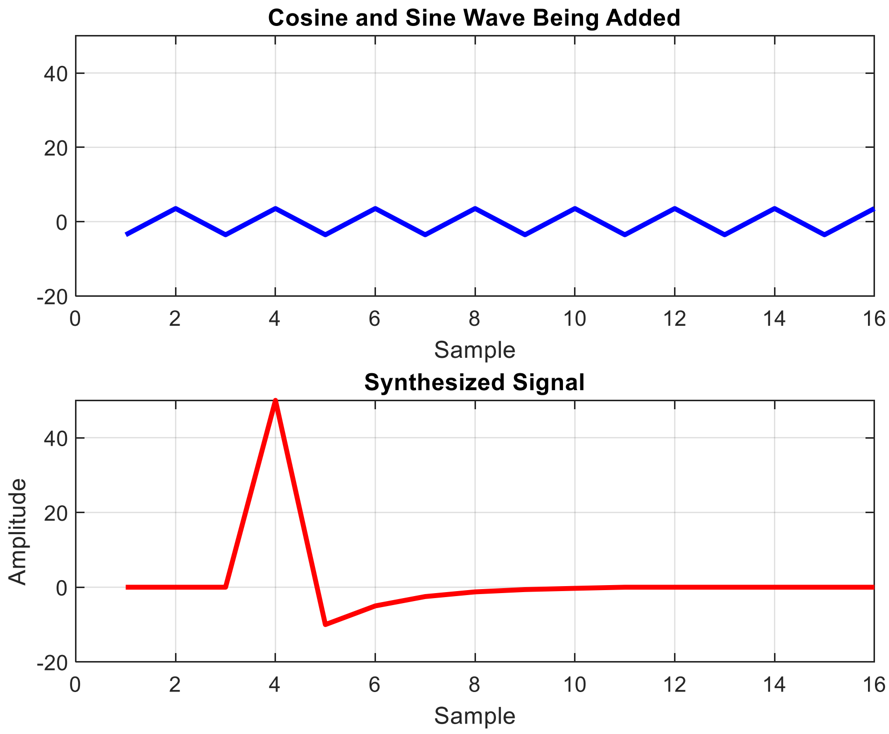
k=8



The Original Signal Synthesized from Each COS and SINE

- The signal has been reproduced exactly from the 9 COS/SINE signal pairs

k=9



DFT ICP

- You have an input sequence [-1,2,3,1]. Compute the DFT. Use the following tables for cos and sine values. Compute $X[0], X[1], X[2]$

			k=0, N=4			k=0, N=4	
i	**x[i]**		**COS(2*pi*k*i/N)**	**SIN(2*pi*k*i/N)**		**x[i] COS(2*pi*k*i/N)**	**x[i] SIN(2*pi*k*i/N)**
0	**-1**		**1**	**0**			
1	**2**		**1**	**0**			
2	**3**		**1**	**0**			
3	**1**		**1**	**0**			
					X[0]		

DFT ICP

			k=1, N=4			k=1, N=4	
i	x[i]		COS(2*pi*k*i/N)	SIN(2*pi*k*i/N)		x[i] COS(2*pi*k*i/N)	x[i] SIN(2*pi*k*i/N)
0	-1		1.000	0.000			
1	2		0.000	1.000			
2	3		-1.000	0.000			
3	1		0.000	-1.000			
					X[1]		

			k=2, N=4			k=2, N=4	
i	x[i]		COS(2*pi*k*i/N)	SIN(2*pi*k*i/N)		x[i] COS(2*pi*k*i/N)	x[i] SIN(2*pi*k*i/N)
0	-1		1.000	0.000			
1	2		-1.000	0.000			
2	3		1.000	0.000			
3	1		-1.000	0.000			
					X[2]		

DFT ICP

			k=0, N=4			k=0, N=4	
i	x[i]		COS(2*pi*k*i/N)	SIN(2*pi*k*i/N)		x[i] COS(2*pi*k*i/N)	x[i] SIN(2*pi*k*i/N)
0	-1		1	0		-1	0
1	2		1	0		2	0
2	3		1	0		3	0
3	1		1	0		1	0
					X[k]	5	0

			k=1, N=4			k=1, N=4	
i	x[i]		COS(2*pi*k*i/N)	SIN(2*pi*k*i/N)		x[i] COS(2*pi*k*i/N)	x[i] SIN(2*pi*k*i/N)
0	-1		1.000	0.000		-1.000	0.000
1	2		0.000	1.000		0.000	2.000
2	3		-1.000	0.000		-3.000	0.000
3	1		0.000	-1.000		0.000	-1.000
					X[k]	-4.000	1.000

NOPRINT

DFT ICP

			k=2, N=4			k=2, N=4	
i	x[i]		COS(2*pi*k*i/N)	SIN(2*pi*k*i/N)		x[i] COS(2*pi*k*i/N)	x[i] SIN(2*pi*k*i/N)
0	-1		1.000	0.000		-1.000	0.000
1	2		-1.000	0.000		-2.000	0.000
2	3		1.000	0.000		3.000	0.000
3	1		-1.000	0.000		-1.000	0.000
					X[k]	-1.000	0.000

$$Re(X[k]) = [5, -4, -1]$$

$$Im(X[k]) = [0, 1, 0]$$

Computing the DFT Values

•

```
280 '
290 '                                   'Correlate XX[ ] with the cosine and sine waves, Eq. 8-4
300 '
310 FOR K% = 0 TO 256                   'K% loops through each sample in REX[ ] and IMX[ ]
320   FOR I% = 0 TO 511    'I% loops through each sample in XX[ ]
330     '
340     REX[K%] = REX[K%] + XX[I%] * COS(2*PI*K%*I%/N%)
350     IMX[K%] = IMX[K%]  - XX[I%] * SIN(2*PI*K%*I%/N%)
360     '
370   NEXT I%
380 NEXT K%
390 '
400 END
```

Values of the DFT

- The values of the DFT are contained in the value of X
  - Magnitudes of COSINE are $Re(\bar{X}[k])$
  - Magnitudes of SINE are $Im(\bar{X}[k])$

- We can also represent each sample in polar format

$$A\cos(x) + B\sin(x) = M\cos(x+\theta) \quad \Rightarrow \quad M\angle\theta$$

Polar Format

Polar Format

- For a point in the DFT

$$Mag(X[k]) = \sqrt{Re(X[k])^2 + Im(X[k])^2}$$

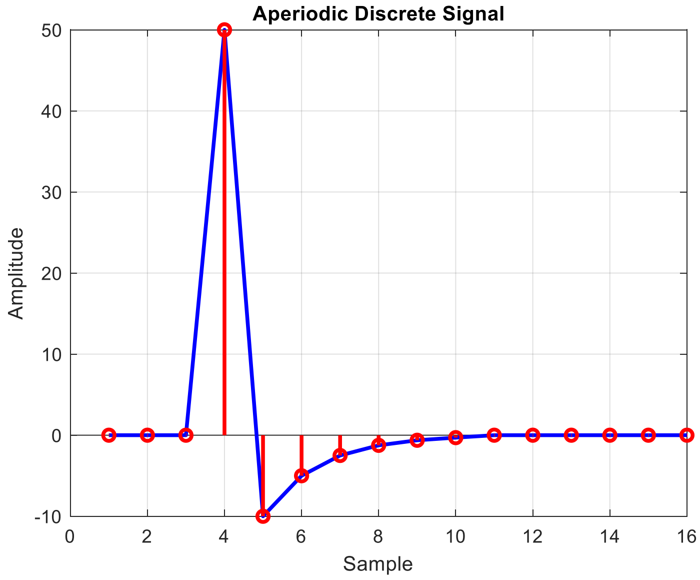
$$Phase(X[k]) = \arctan\left(\frac{Im(\mathrm{X[k]})}{Re(\mathrm{X[k]})}\right)$$

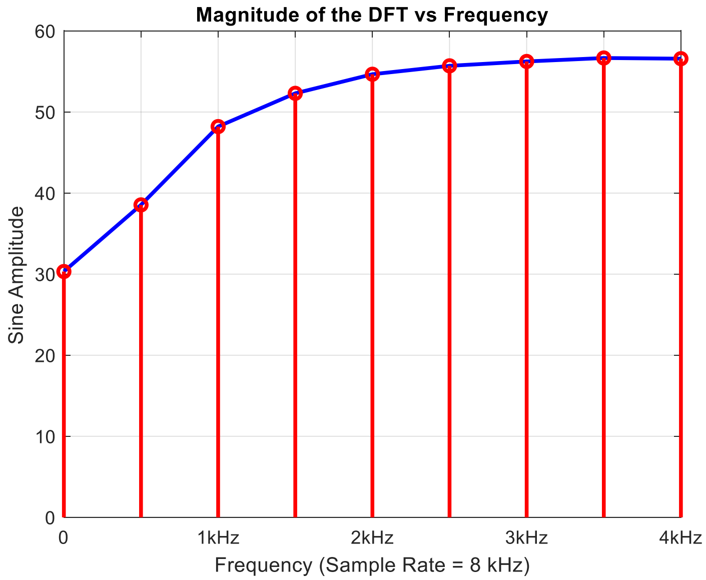
Polar Format

- Polar format allows us to think of the DFT in two ways
  - An N point signal decomposed into $N/2+1$ cosine and sine waves
  - An N point signal decomposed into $N/2+1$ cosine waves with a magnitude and a phase

Polar Format

- Use of the polar format allows us to look at the magnitude and phase of a signal in the frequency domain





Polar Format

- Use of the polar format allows us to look at the magnitude and phase of a signal in the frequency domain