Digital Signal Processing

Linear Systems

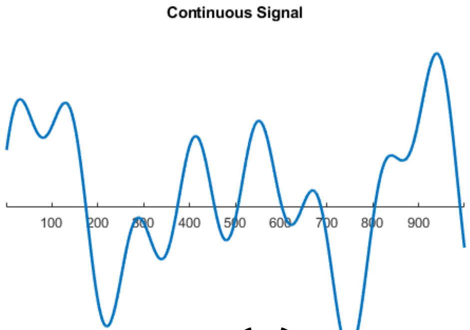
Today's Topics

- Signals and Systems
- Linearity, Homogeneity, Shift Invariance
- Superposition
- In Class Problem
- Summary

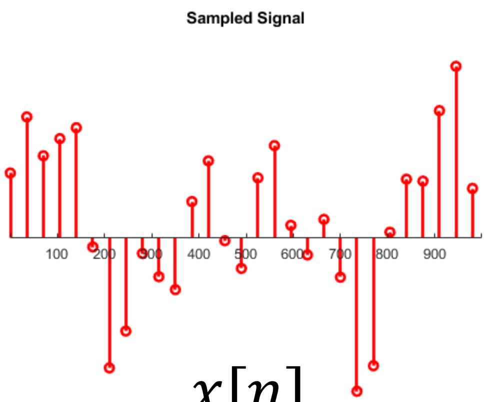
Signals and Systems

- A signal is a description of how one parameter varies with another.
  - Example – Voltage vs Time

- Notation Convention
  - Signals use lower case variables
  - Continuous time uses (t) t is time
  - Discrete time uses $[n]$ n is sample index



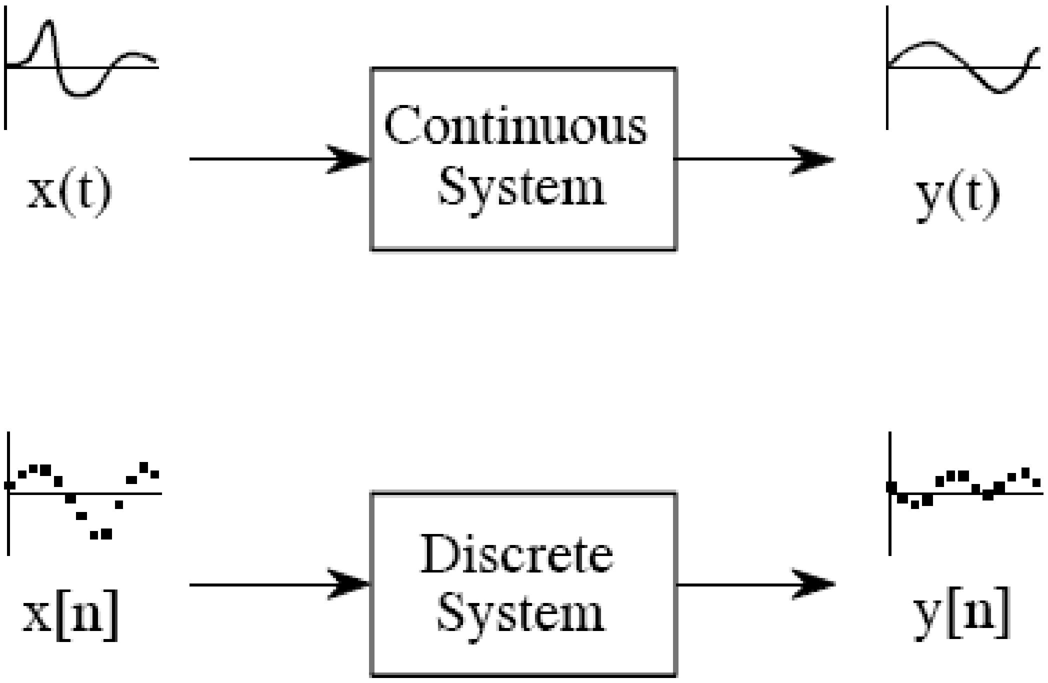
$x(t)$



$x[n]$

Signals and Systems

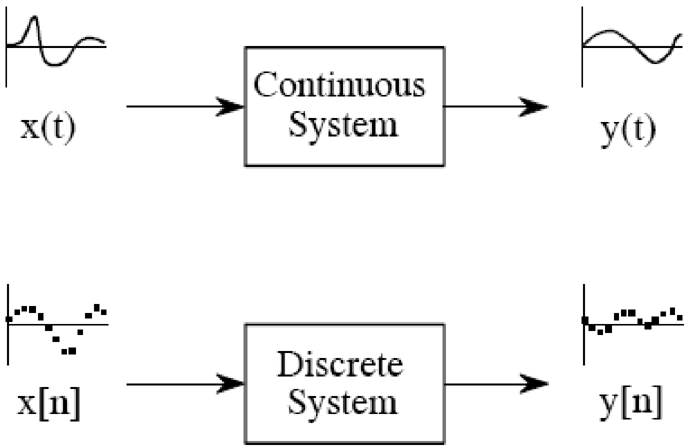
- A system is any process that produces an output signal in response to an input signal.



System Design or Analysis

- You may have to design a system that produces a desired output for a given input
  - What system will create output $y[n]$ for an input $x[n]$

- You may have to analyze a system
  - What will be the output $y[n]$ for a given $x[n]$ as input



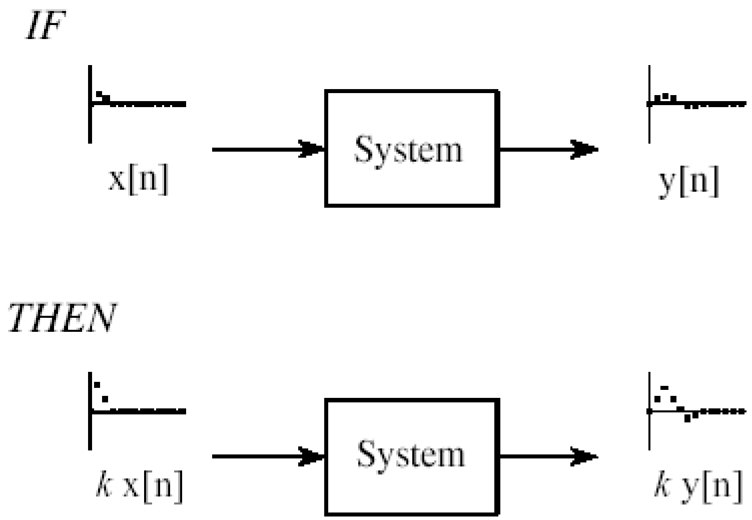
System Linearity

- For a system to be linear, must have two properties
  - ***Homogeneity*** and ***additivity***.
  - If a system has both then it is linear.

- Another property critical for DSP, but not for linearity is time shift invariance
  - The system responds in the same way to a signal regardless of time

Homogeneity

- Scaling the input signal by a scale factor K causes the output to scale the same factor K.



- y(t) =x(t)*x(t) fails the test of homogeneity

Simple Homogeneity Example

- My system is an amplifier with a gain of 5

System Equation $\rightarrow y = f(x) = 5x$

For an input $x_1 = 2$

$$y_1 = f(x_1) = 5(x_1) = 5(2) = 10$$

For an input $x_2 = 2x_1 = 4$

$$y_2 = f(x_2) = 5(x_2) = 5(4) = 20$$

$$y_1 = f(x_1) = 10 \quad and \quad y_2 = f(2x_1) = 20 = 2(y_1)$$

A Non-Homogenous System

- My system is described by the equation

$$\text{System Equation} \rightarrow y = f(x) = x \times x = x^2$$

$$\text{For an input } x_1 = 2$$
$$y_1 = f(x_1) = x_1^2 = 4$$

$$\text{For an input } x_2 = 2x_1 = 4$$
$$y_2 = f(x_2) = x_2^2$$
$$\mathrm{y}_2 = 4^2 = 16$$

$$y_1 = f(x_1) = 4 \quad y_2 = f(2x_1) = 16$$
$$y_2 \neq 2y_1$$

Additivity

- If 2 (or more) input signals added together pass through the system without interacting, then the system is additive.

- Mathematically:

System Equation: $y = f(x)$

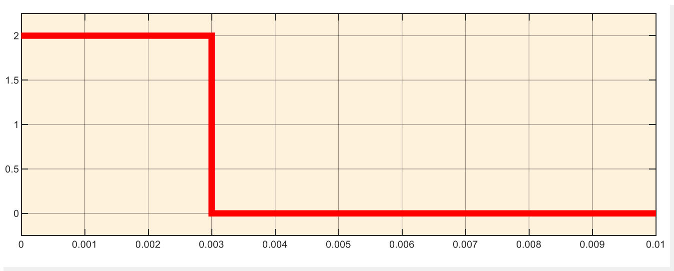
$$y_1 = f(x_1) \qquad\qquad y_2 = f(x_2)$$

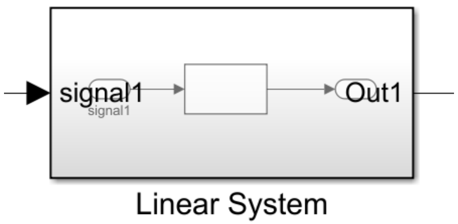
If : $y_{12} = f(x_1) + f(x_2) = f(x_1 + x_2)$

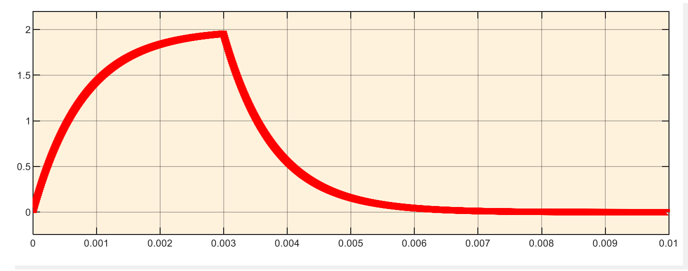
Then the system is additive

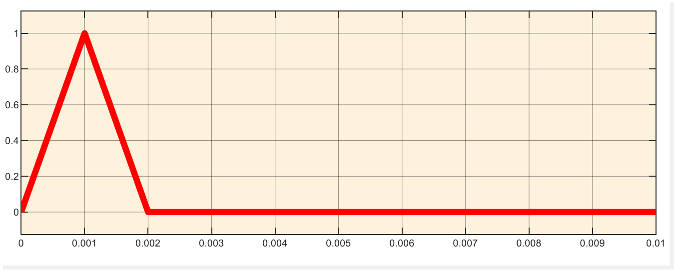
Additivity MATLAB Demo

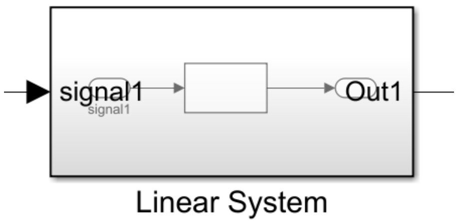
- Take a square pulse and a triangle pulse. Put them individually through a linear system (a filter)

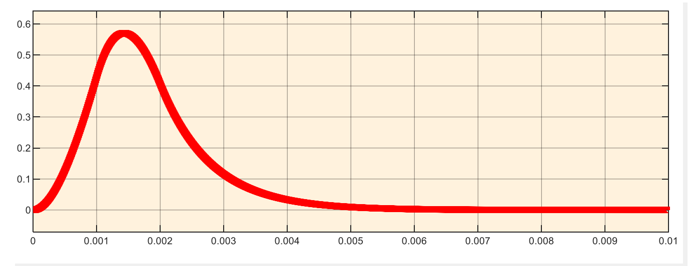






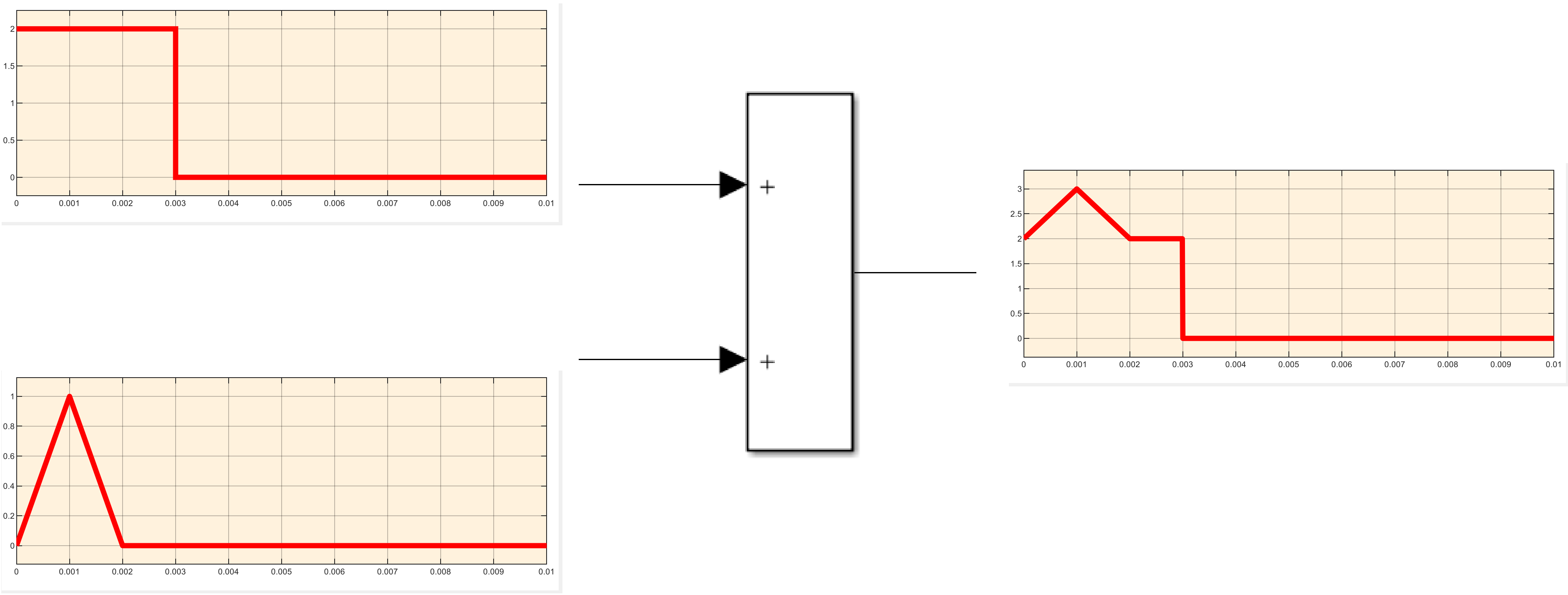




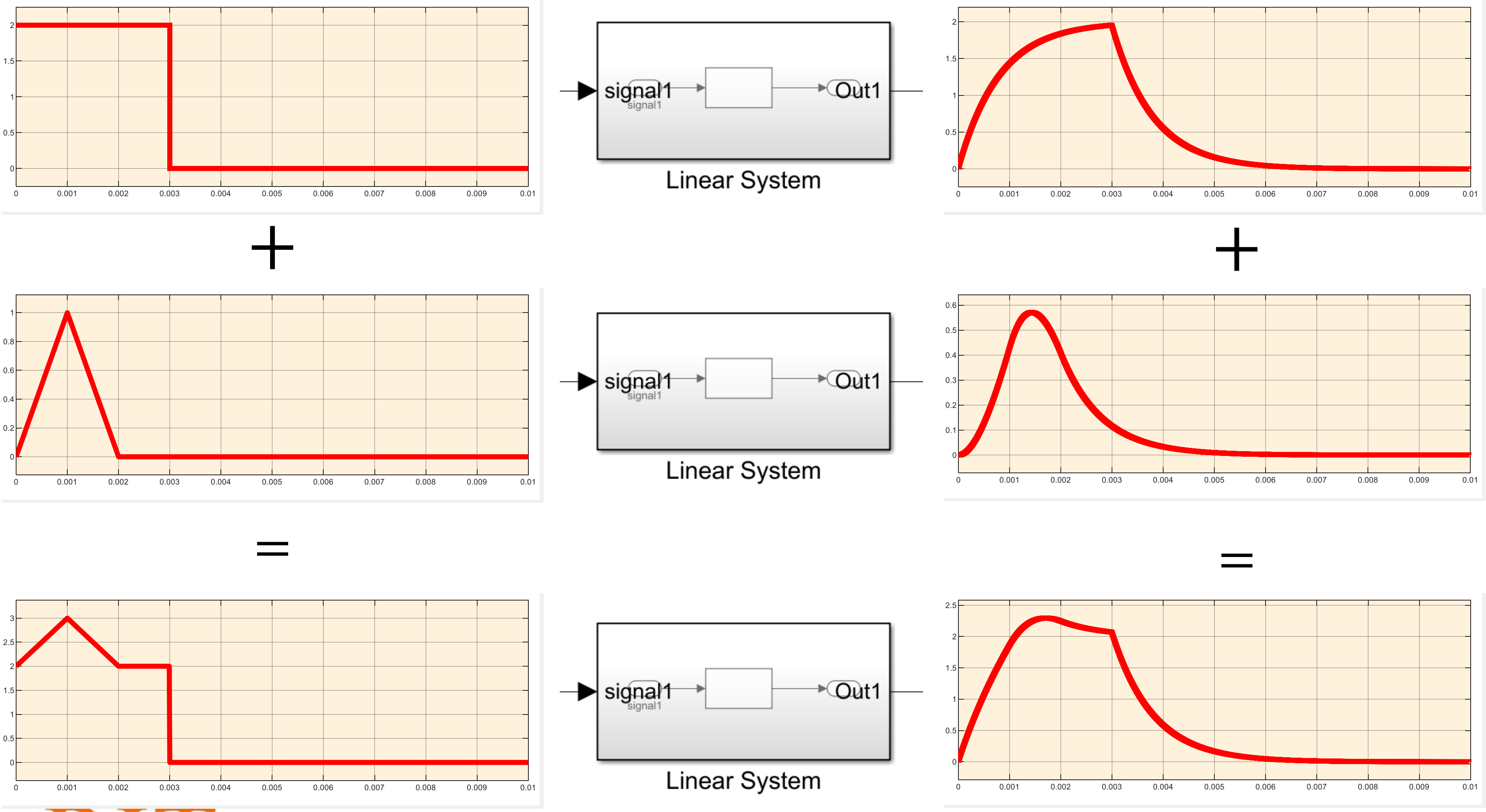


Additivity MATLAB Demo

- Add the two pulses and put the results through the system



Additivity MATLAB Demo



Additivity

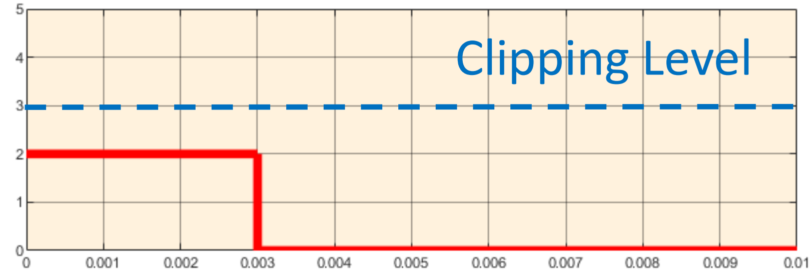
- One situation where non-additivity occurs is during saturation.

- If the combined signals exceed a limit and are clipped then the system becomes non-additive

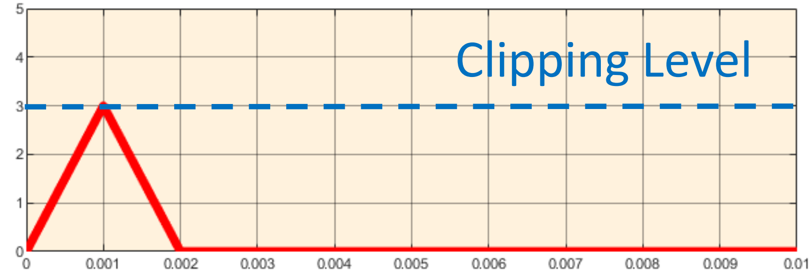
- This might happen if the ADC input value is exceeded or an operation causes a value to go out of range (e.g. a fixed point roll over)

Case of "Clipping" the Signals When Combined

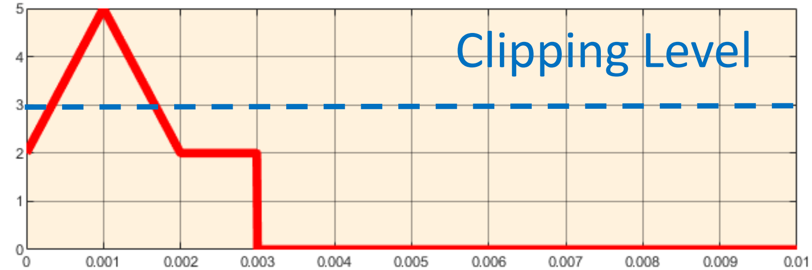
Pulse Input
$V_{in} < 3V$



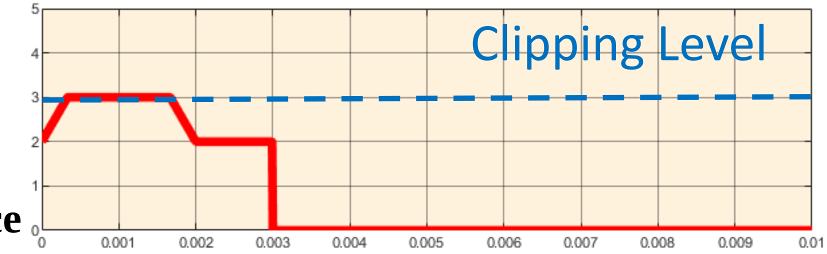
Triangle Input
$V_{in} < 3V$



Summed Signals
$V_{in} > 3V$



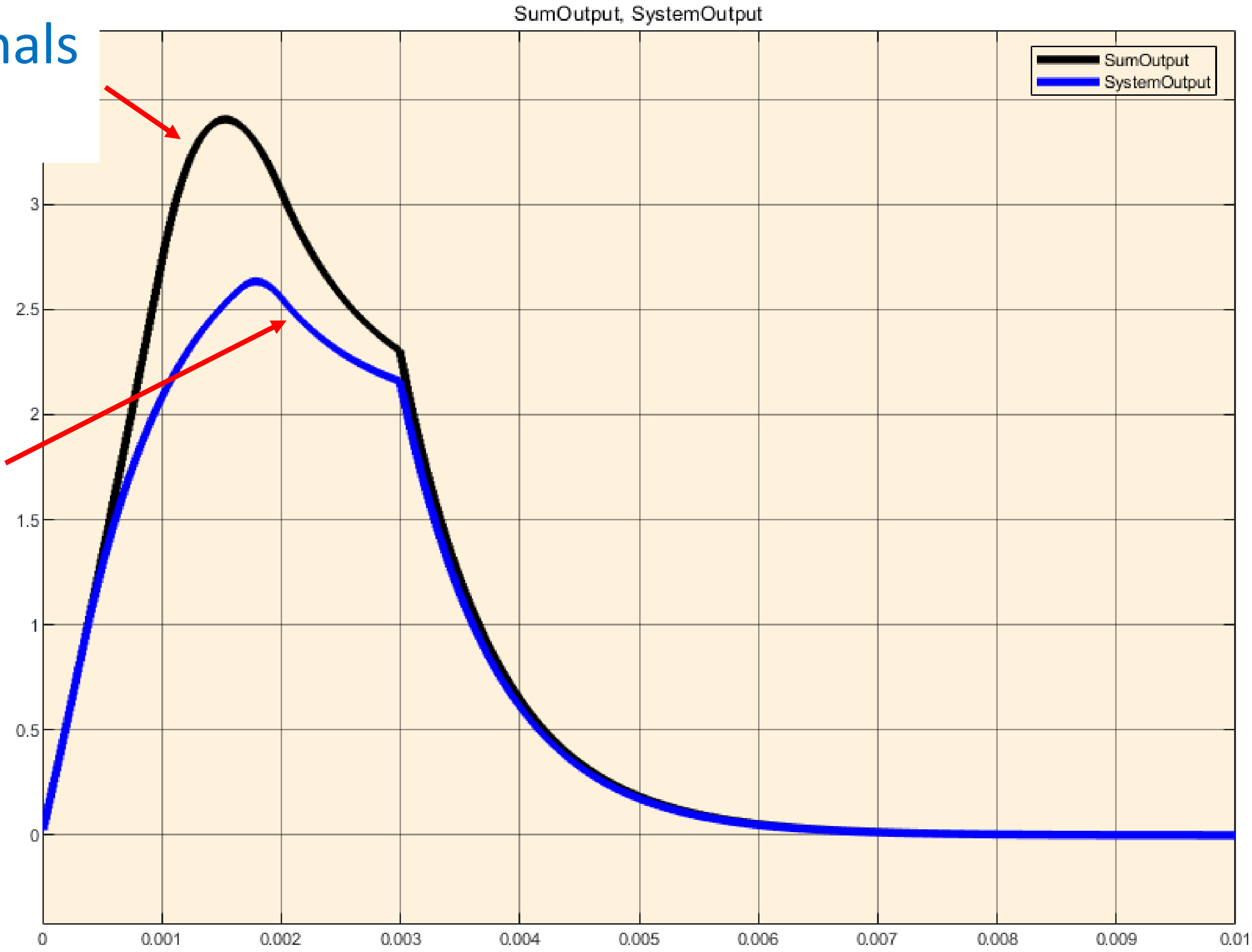
Clipped Signal



Output of System with Individual Signal and "Clipped" Signal

Individual Signals Output

Clipped Signal Output



Time Shift Invariance

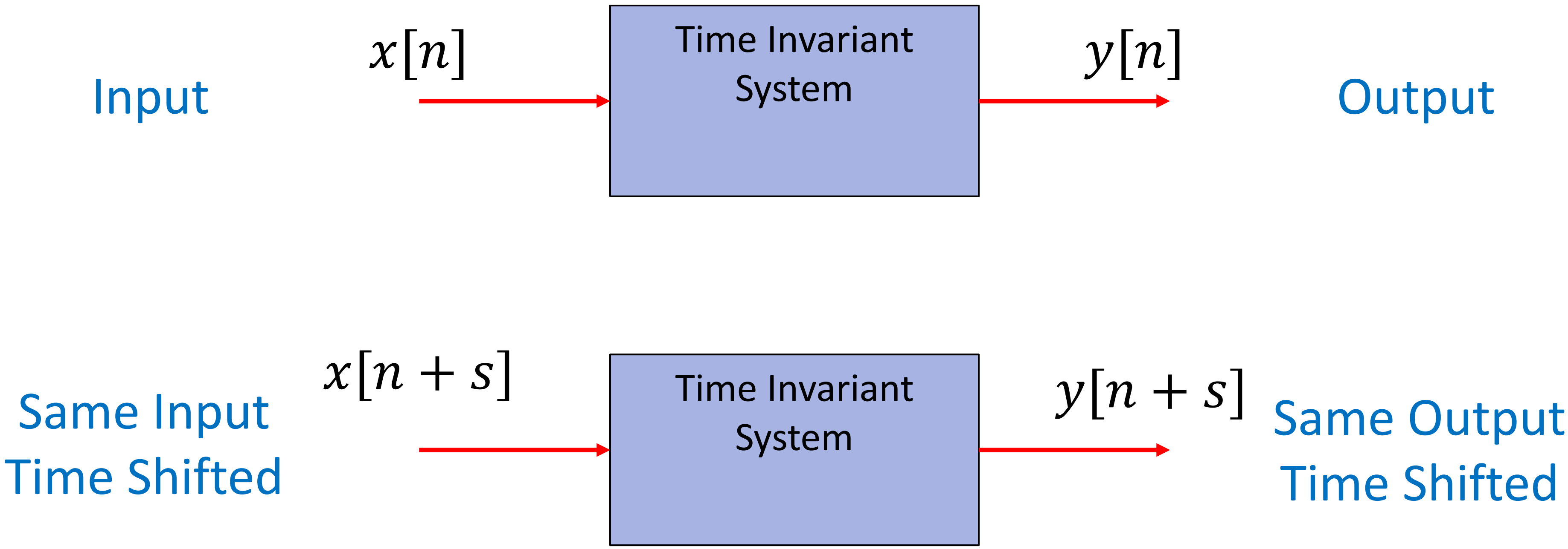
- A system is time shift invariant if a time shift in the input signal causes an identical shift in the output signal.

- Mathematically
  - If $x(t)$ produces $y(t)$ then $x(t + s)$ produces $y(t + s)$ – For a continuous system

  - If $x[n]$ produces $y[n]$, then $x[n + s]$ produces $y[n + s]$ -- For a discrete time system

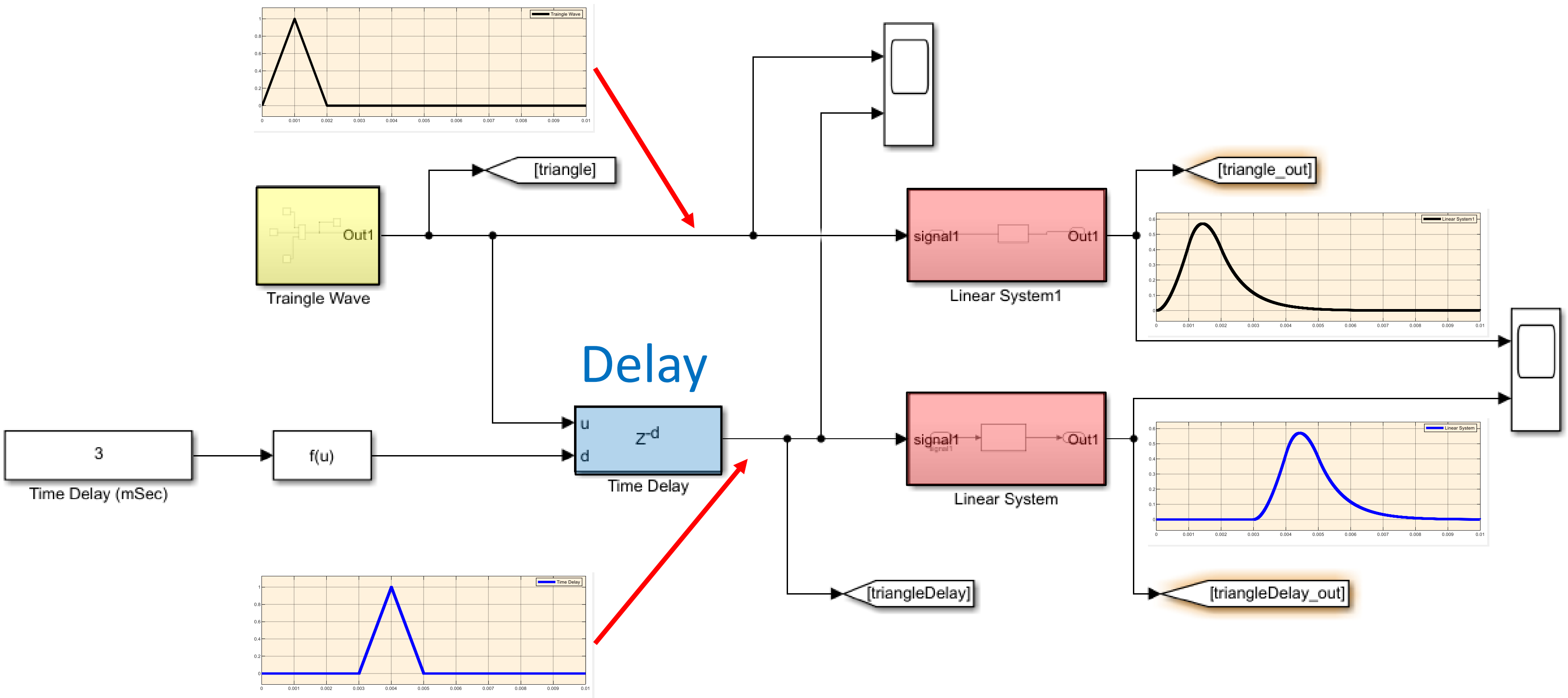
Time Shift Invariance

- The same input but time shifted produces the same output but time shifted



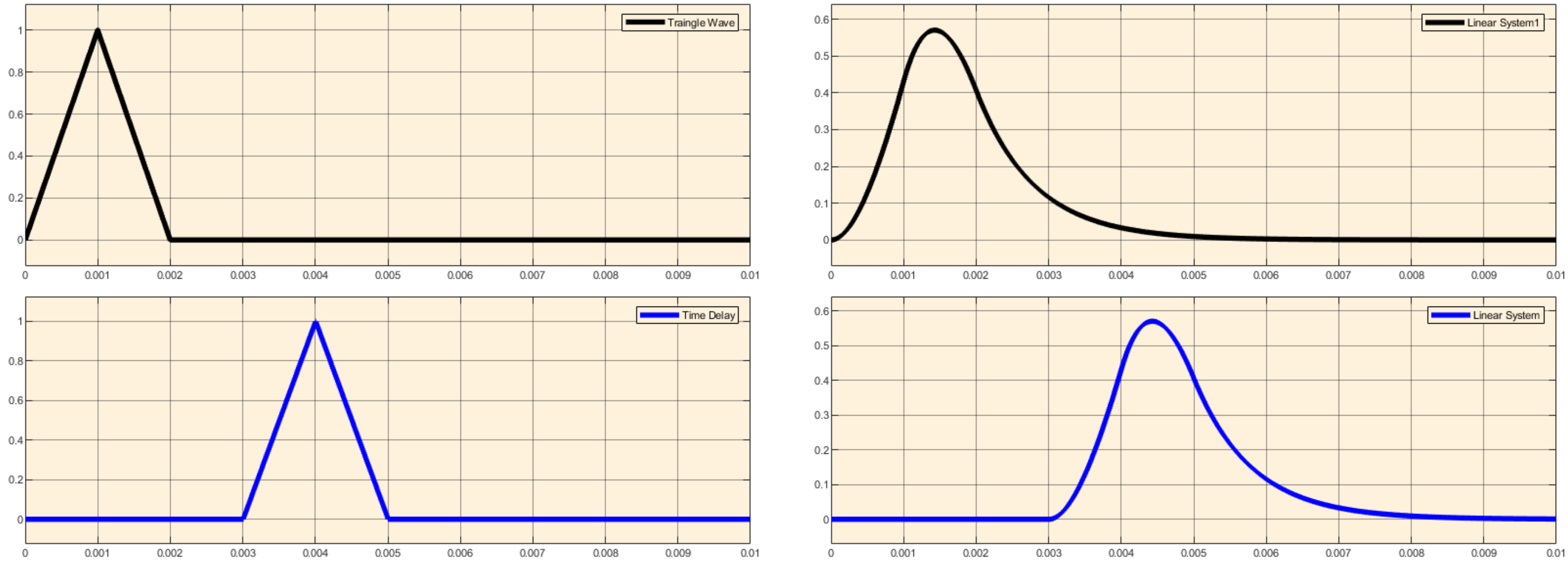
Time Shift Invariance Simulink Example

- Take the input triangle wave and time shift it
- Apply both signals to the same linear system



Time Shift Invariance

- A time shift in the signal does not change how the system responds.
- The same output signal just shifted in time



Sinusoidal Fidelity

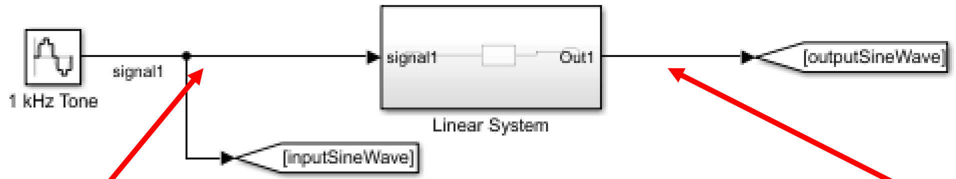
- If the input to a linear system is a sinusoidal wave then the output will also be a sinusoidal wave at the same frequency as the input.
- The output may have different phase and amplitude with respect to the original input waveform

Sinusoidal Fidelity
Example – A Low Pass Filter

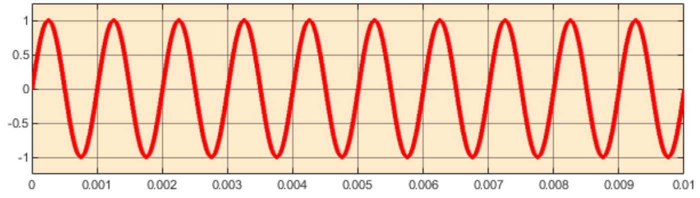
- Sine wave in. Sine wave out. Same Frequency
- Possibly different amplitude and phase output

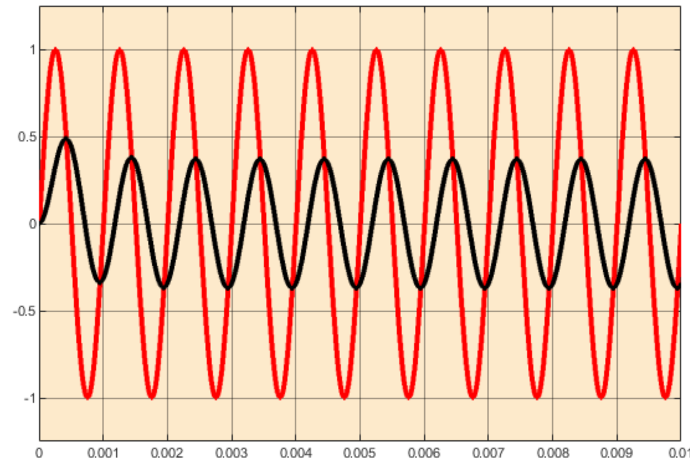
400 Hz LPF

1 kHz Sine



Amplitude Change
Phase Shifted

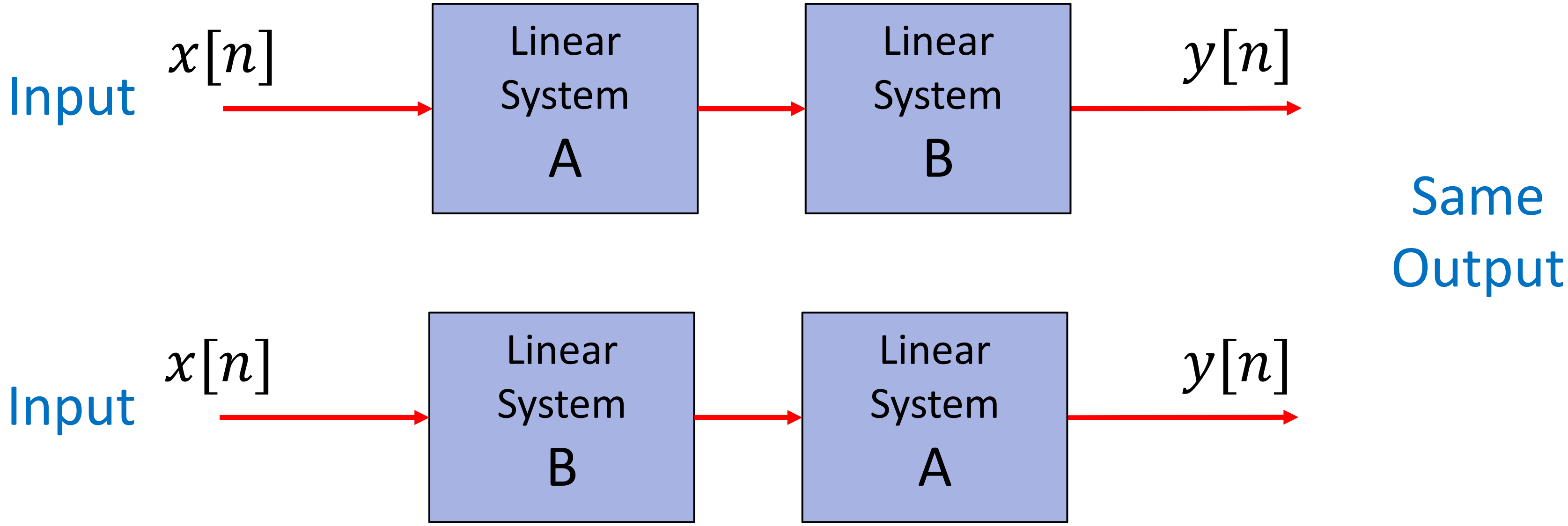




Input and Output

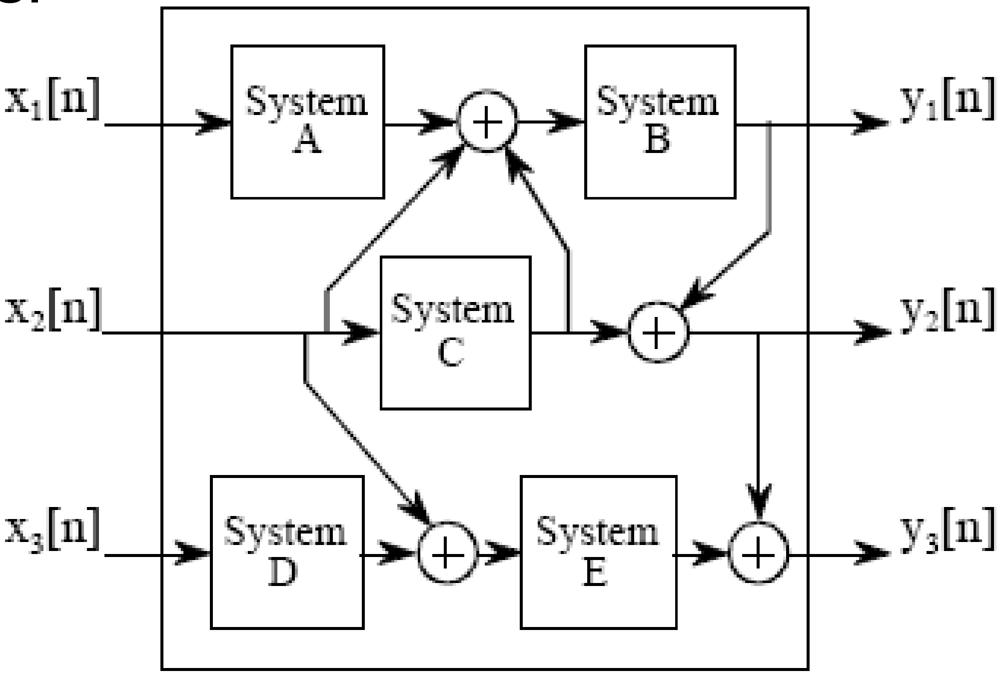
Properties of Linearity Commutative

- Commutative - the order/sequence of two cascaded linear systems does not change the resulting output of the system.



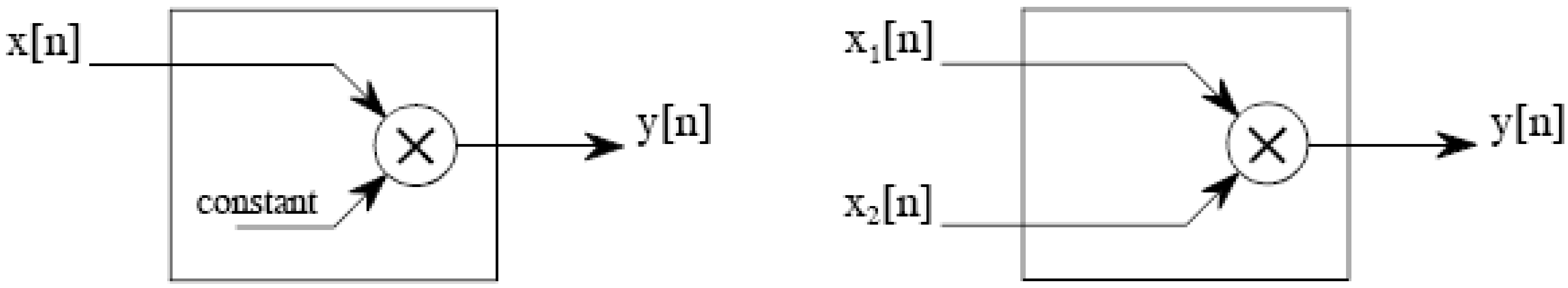
Multi-Input Systems

- The multi-input multi-output system can be arbitrarily complex but as long as each internal system is linear, the output can be determined by superposition of responses.



Linearity of Multiplication

- Multiplication is linear when a signal is multiplied by a constant
- Multiplication is non-linear when two signals are multiplied by each other.



Linear
a. Multiplication by a constant

Nonlinear
b. Multiplication of two signals

Synthesis / Decomposition

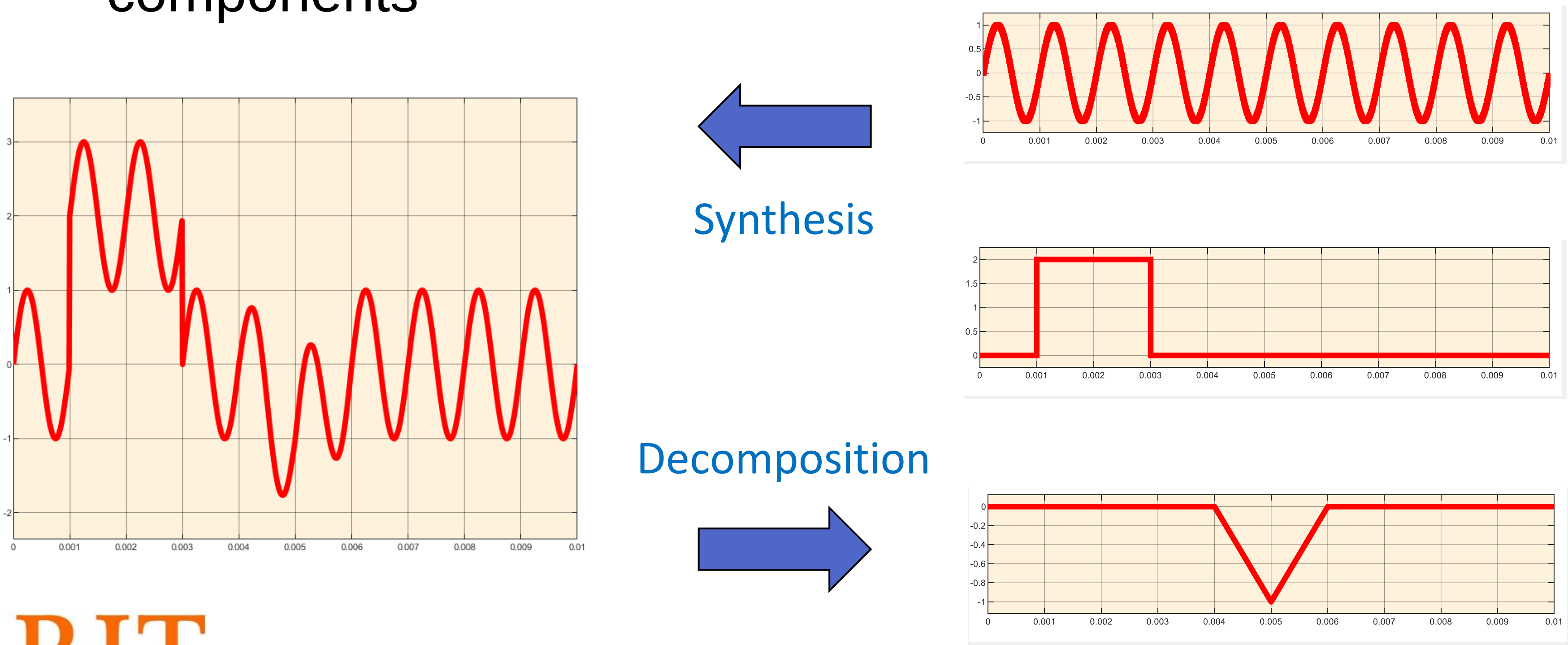
- Signals may be <u>synthesized</u> by scaling and adding individual signals.

$$y = c_1(x_1) + c_2(x_2)$$

- Breaking a signal down into two or more additive components is called <u>decomposition</u>.

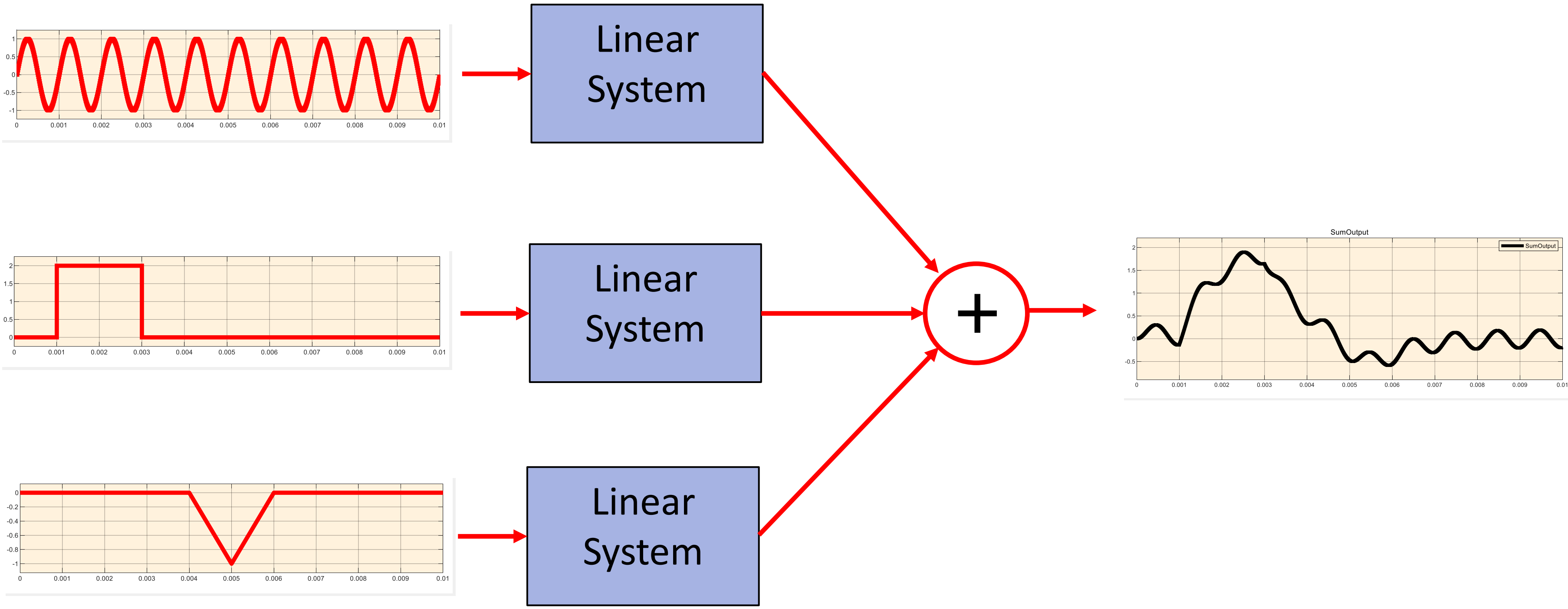
Signal
Synthesis and Decomposition

- Synthesis – Adding two or more signals to create a new signal
- Decomposition – Breaking down a signal into additive components



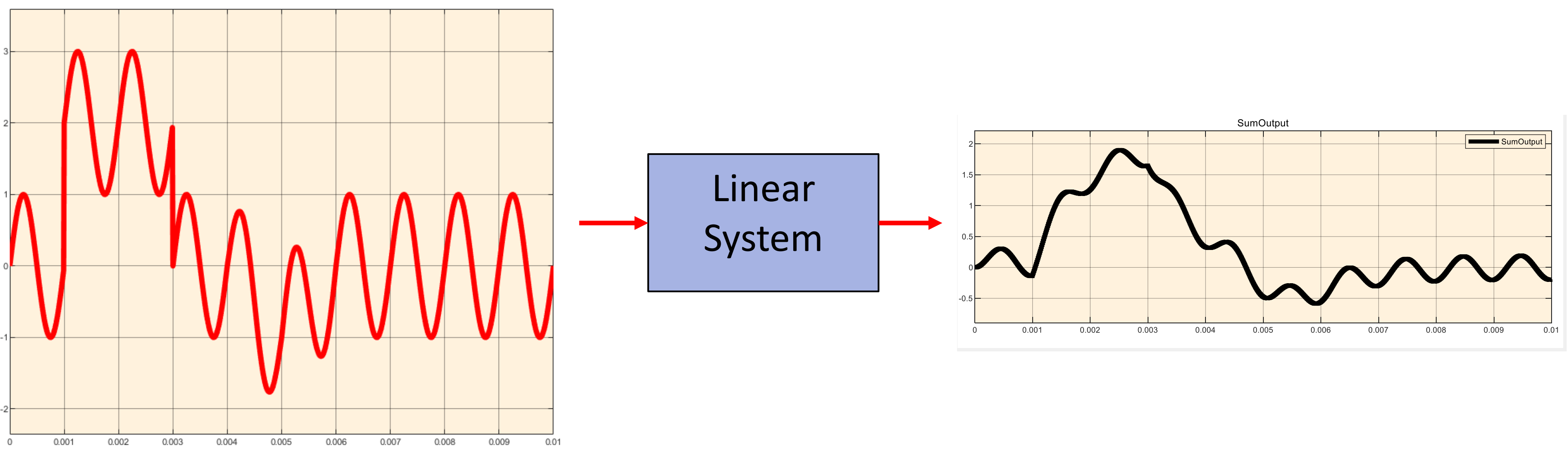
Fundamental Property of DSP

- If I decompose a signal and apply each part to a linear system then recombine…



Fundamental Property of DSP

- The output will be the same as if I applied the entire signal to the linear system



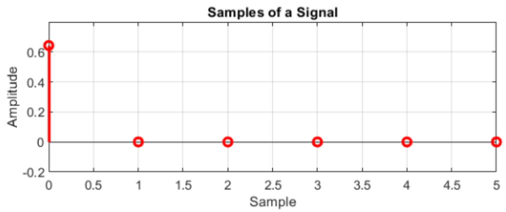
Various Types of Decomposition

- Impulse Decomposition – Breaking into impulses
- Step Decomposition
- Even and Odd Function Decomposition
- Interlaced Decomposition
- Fourier Decomposition

Impulse Decomposition

- What if we decompose the signal into impulses at each sample

1



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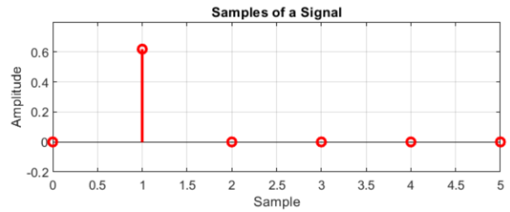

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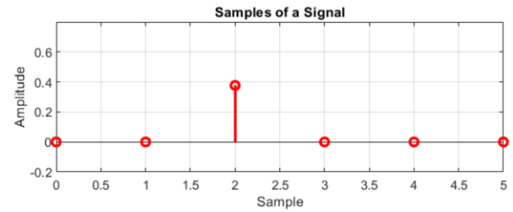
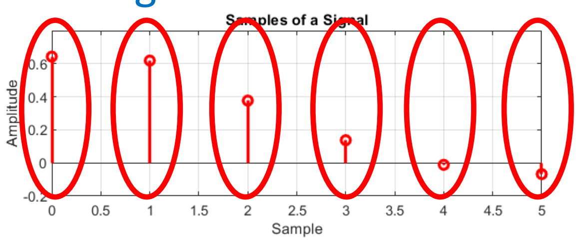


Full Signal



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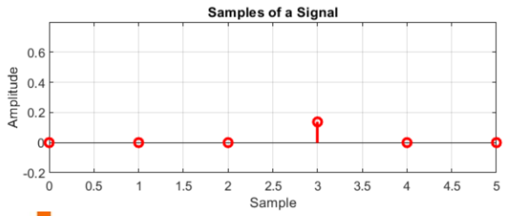

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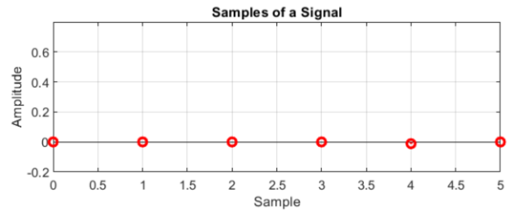

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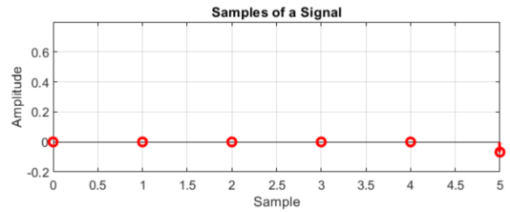
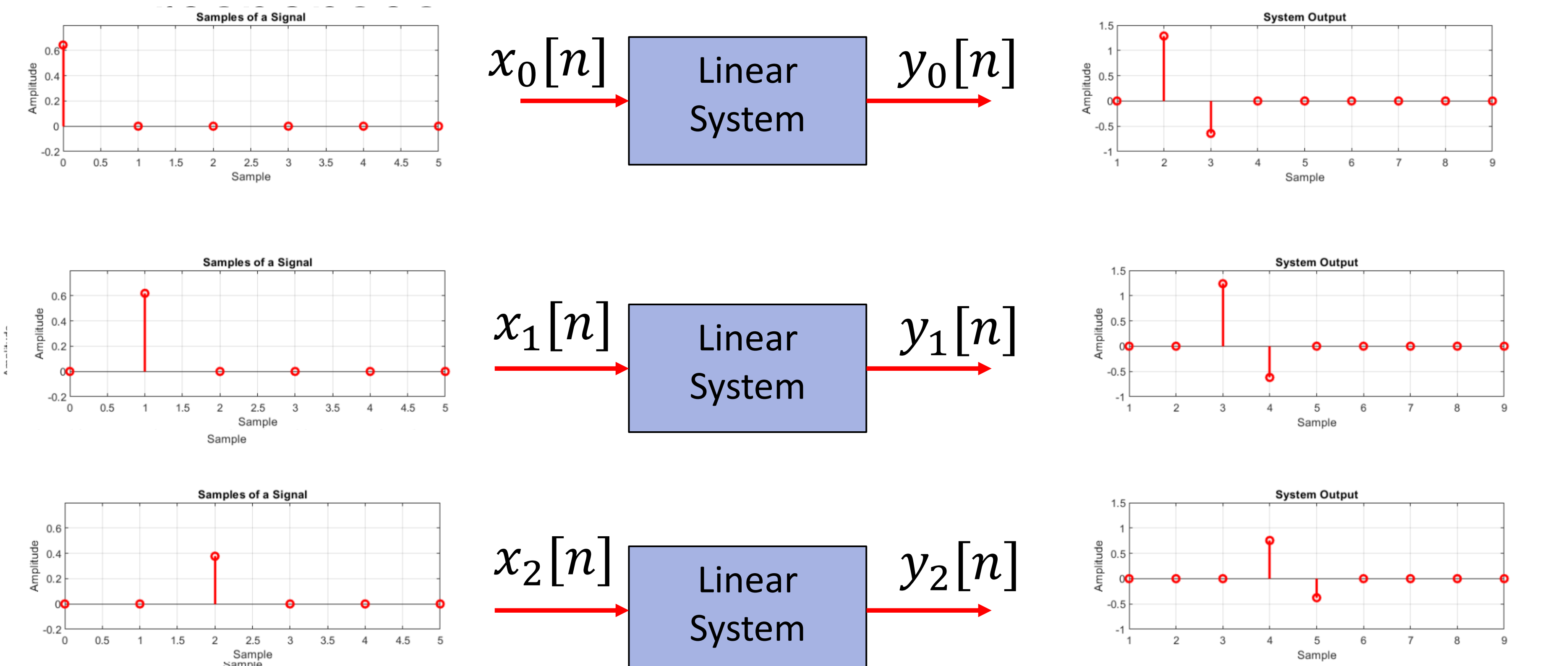


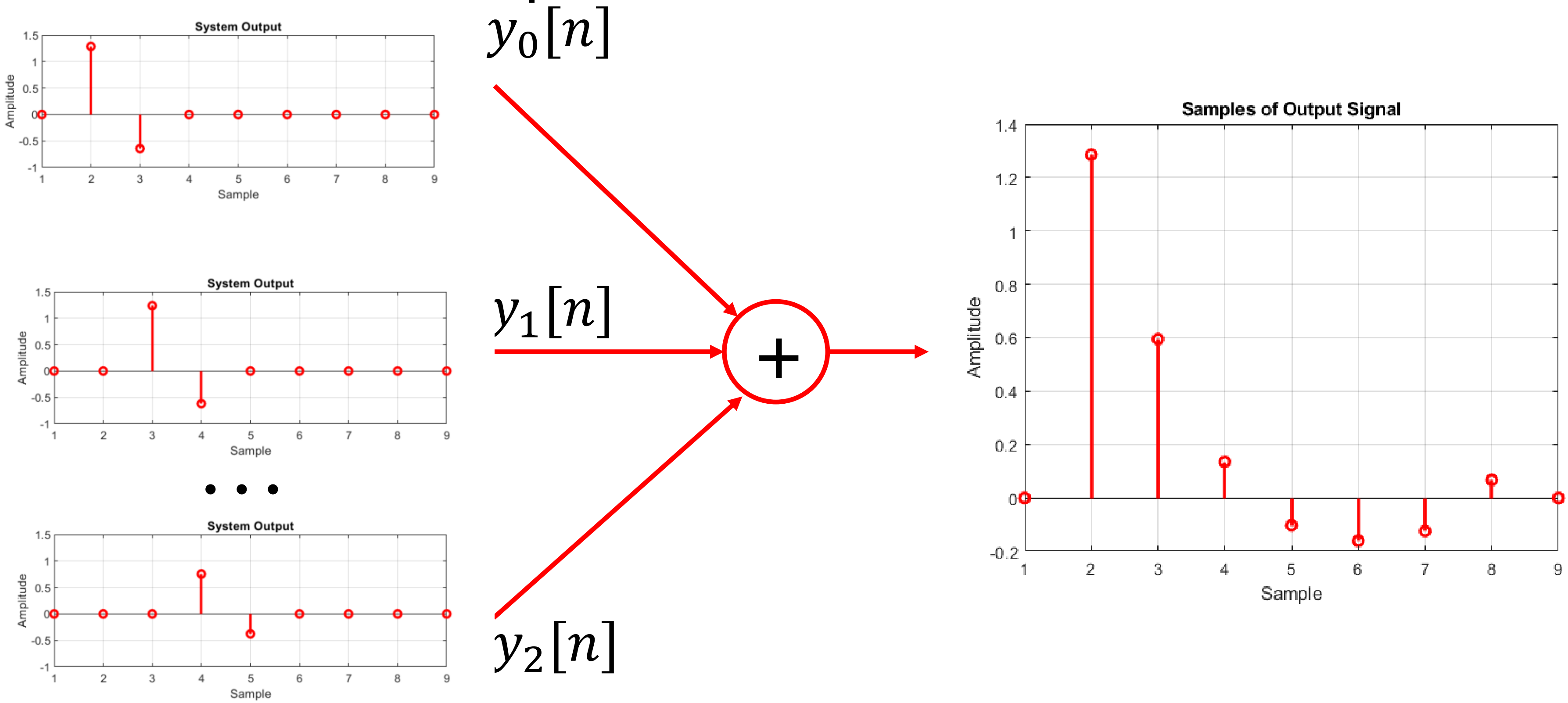
Impulse Decomposition

- Pass each individual input signal (3 shown) through the system producing a set of output



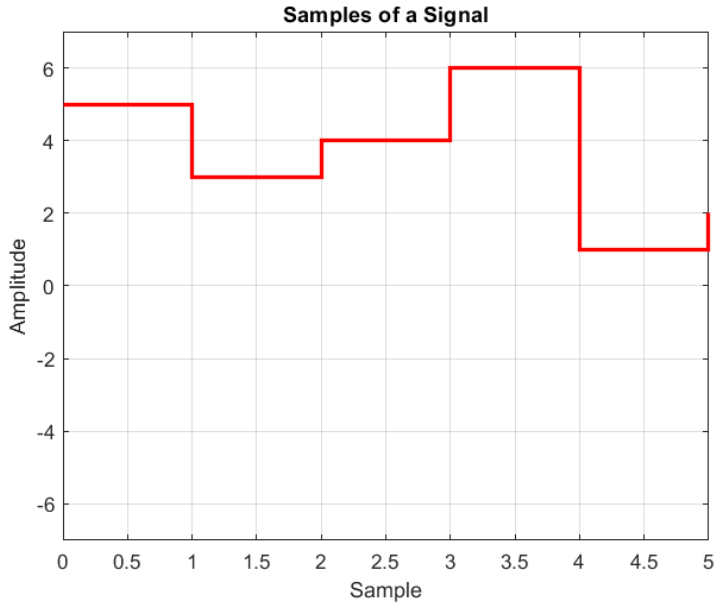
Impulse Decomposition

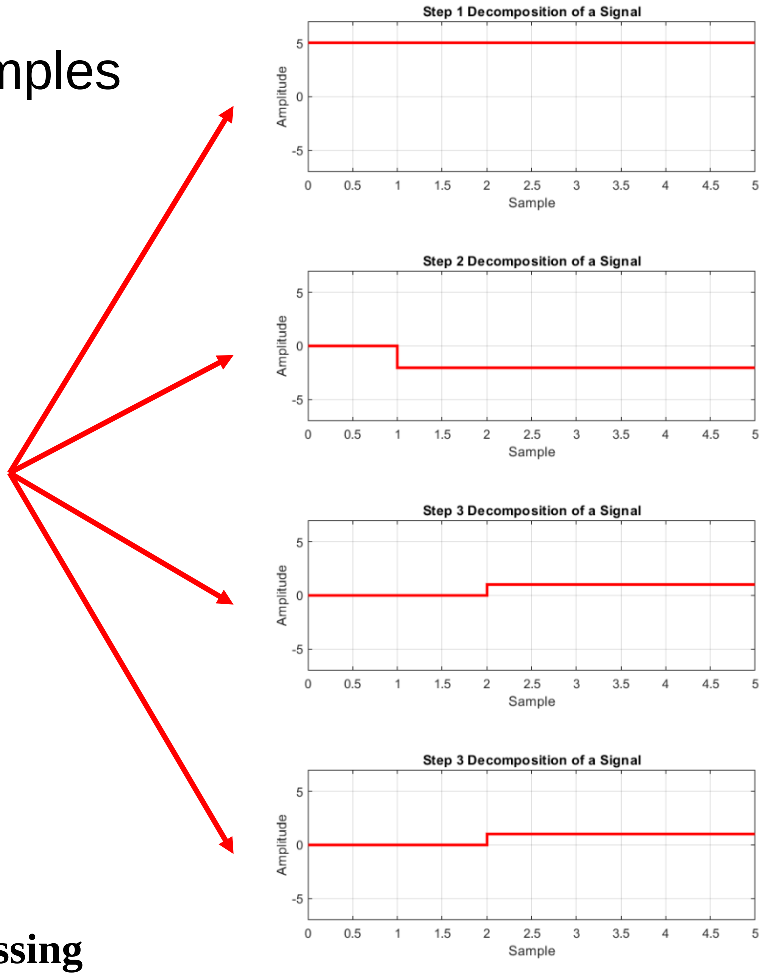
- Combining individual output responses to get the final output



Step Decomposition

- Take N samples of the waveform and break them into step functions
- Each step is the Δ between samples





Even/Odd Function Decomposition

- Even symmetry – Samples mirrored around the center

$$x[N/2+1] = x[N/2-1]$$

- Odd Symmetry -- Mirrored around center, but values are inverted in sign

$$x[N/2+1] = -x[N/2-1]$$

- Assumes an even number of samples running from 0 to N-1

Even/Odd Function Decomposition

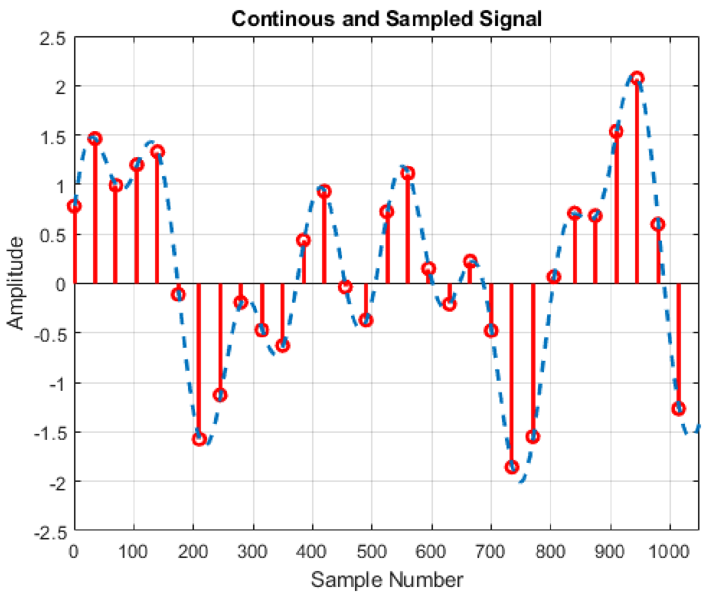
- Compute the two functions using

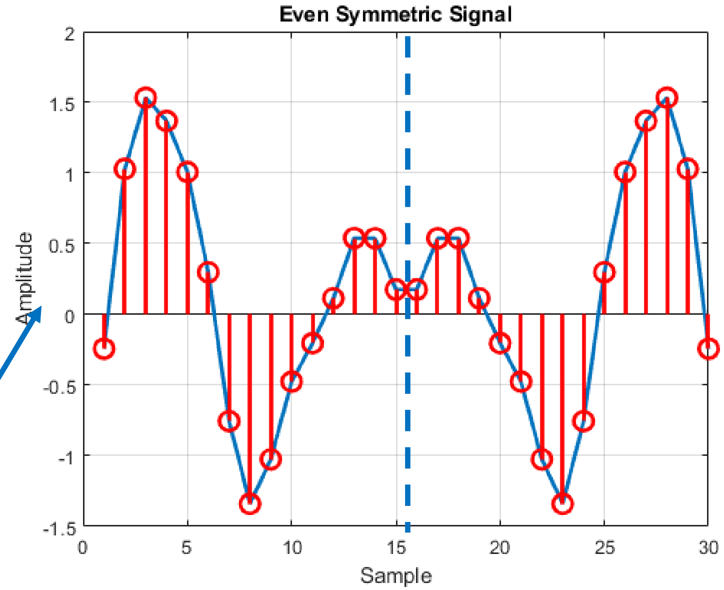
$$x_{even}[n] = \frac{x[n] + x[N-1-n]}{2}$$

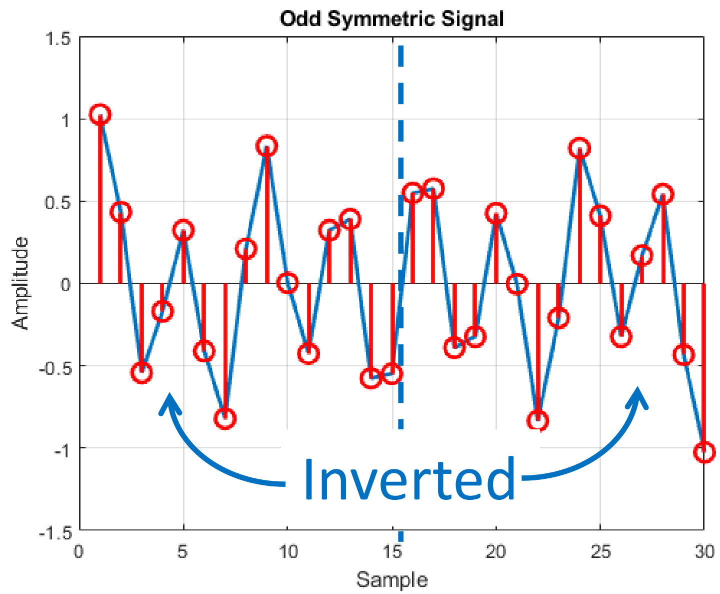
$$x_{odd}[n] = \frac{x[n] - x[N-1-n]}{2}$$

- Assumes an even number of samples running from 0 to N-1

Even and Odd Symmetry







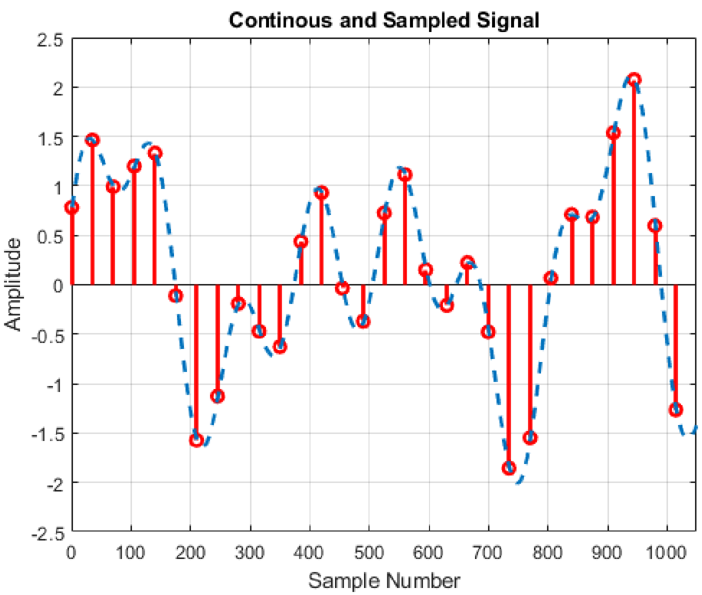
- Decompose the signal into even and odd symmetric signals

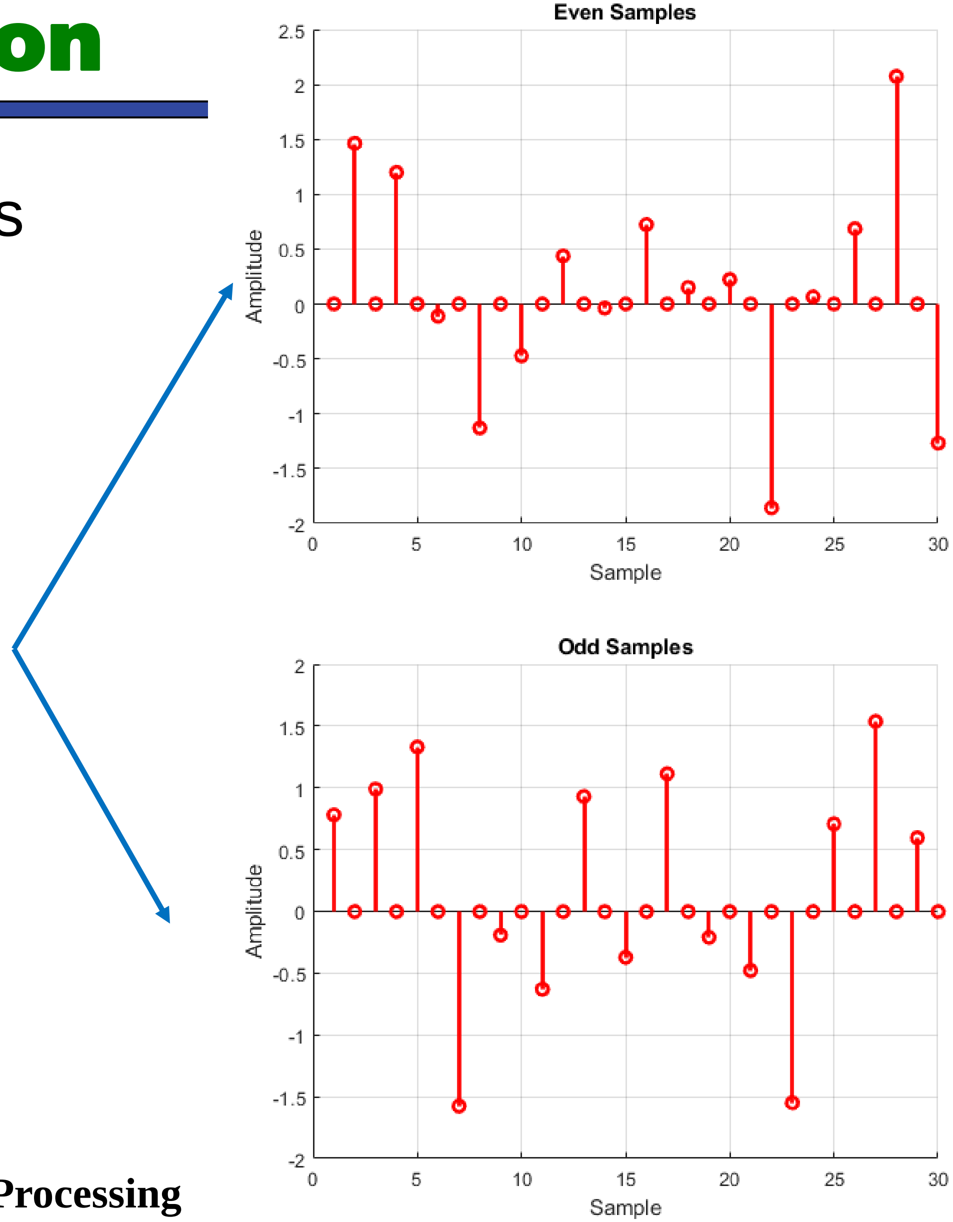
Interlaced Decomposition

- Decompose signal
  - Even numbered samples
  - Odd numbered samples
- The non-odd /non-even sample number values are set to zero.
- This decomposition is used in Fast Fourier Transform FFT algorithm

Interlaced Decomposition

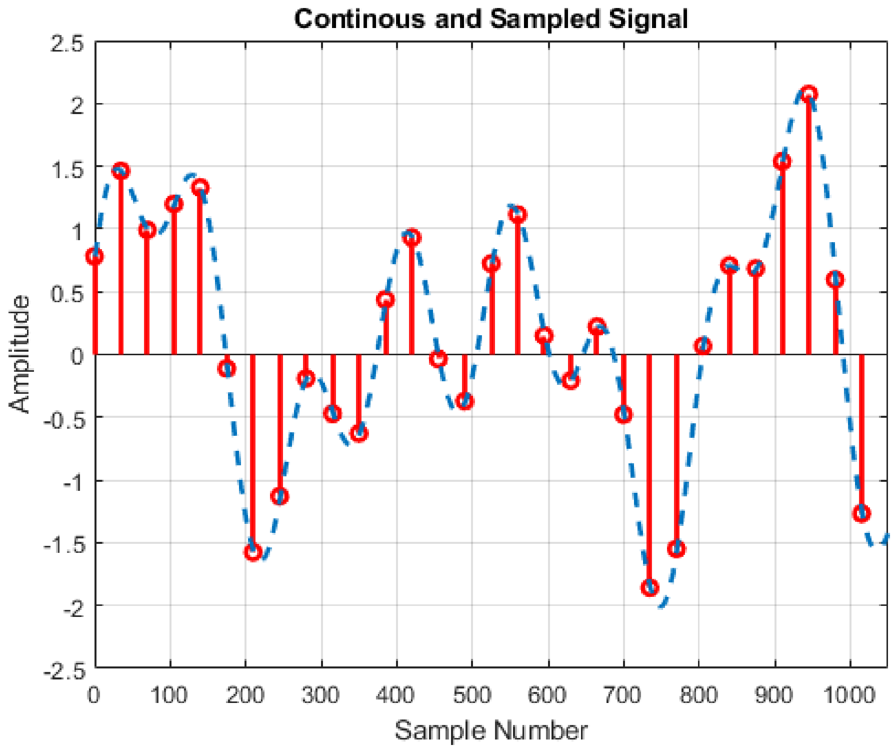
- Even and Odd Samples Numbers



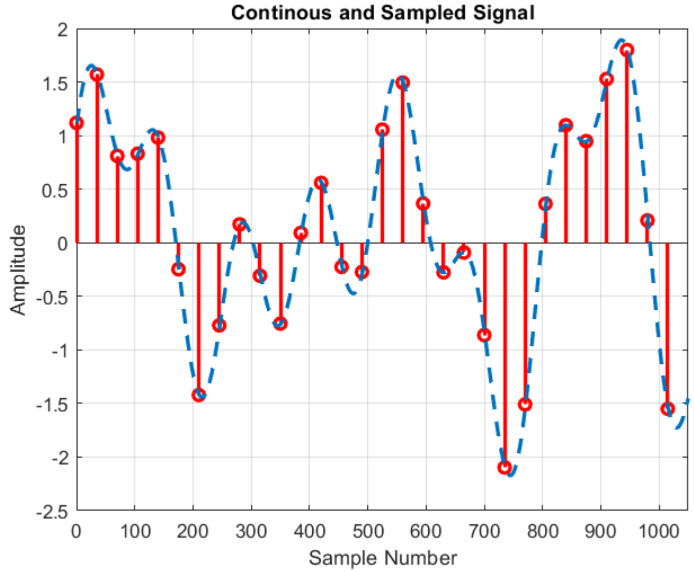


Fourier Decomposition

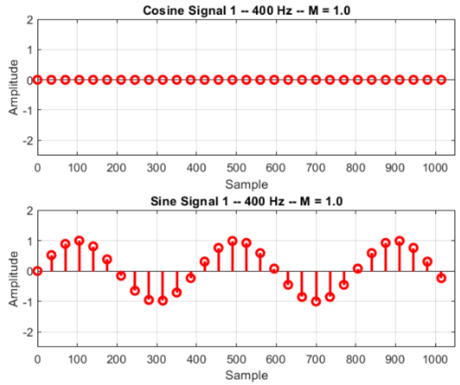
- Decompose the signal into a set of SINE and COSINE waves at different frequencies



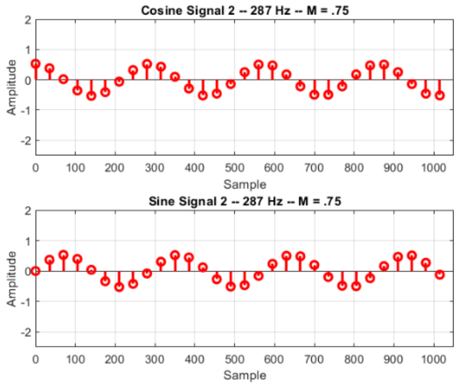
$$cos(\omega t + B) = \sin(B)\cos(\omega t) + \cos(B)\sin(\omega t)$$



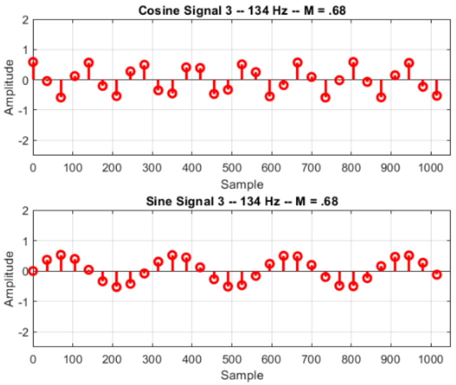
400 Hz M = 1.0



287 Hz M = .75



134 Hz M = .68



In Class Problem

- Two discrete waveforms each 8 samples long
  - $x[n] = 1, 2, 3, 4, -4, -3, -2, -1$
  - $y[n] = 0, -1, 0, 1, 0, -1, 0, 1$
- Sketch $x[n-1]$ and $y[n+3]$ (add zeros where necessary)
- Sketch even and odd interlaced sample decompositions of $x[n]$
- Sketch the even symmetry decomposition of $y[n]$

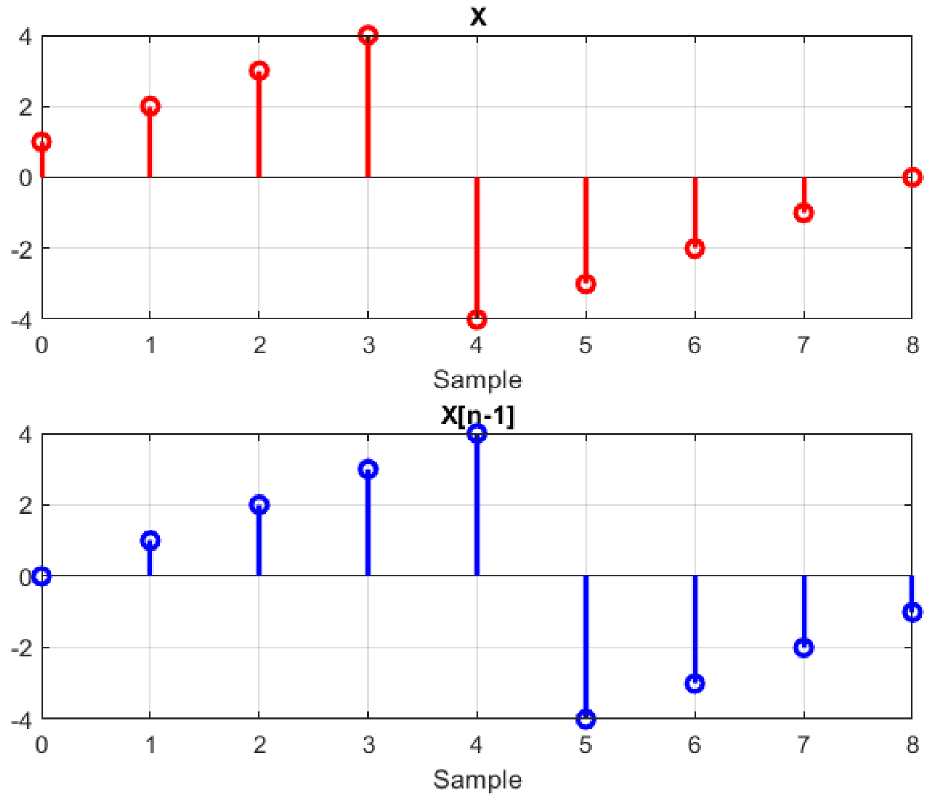
In Class Problem
Time Shifting

- $x[n-1]$ is the waveform x shifted right by 1 sample

$$x[n] = 1, 2, 3, 4, -4, -3, -2, -1, 0$$

$$\Downarrow$$

$$x[n-1] = 0, 1, 2, 3, 4, -4, -3, -2, -1$$

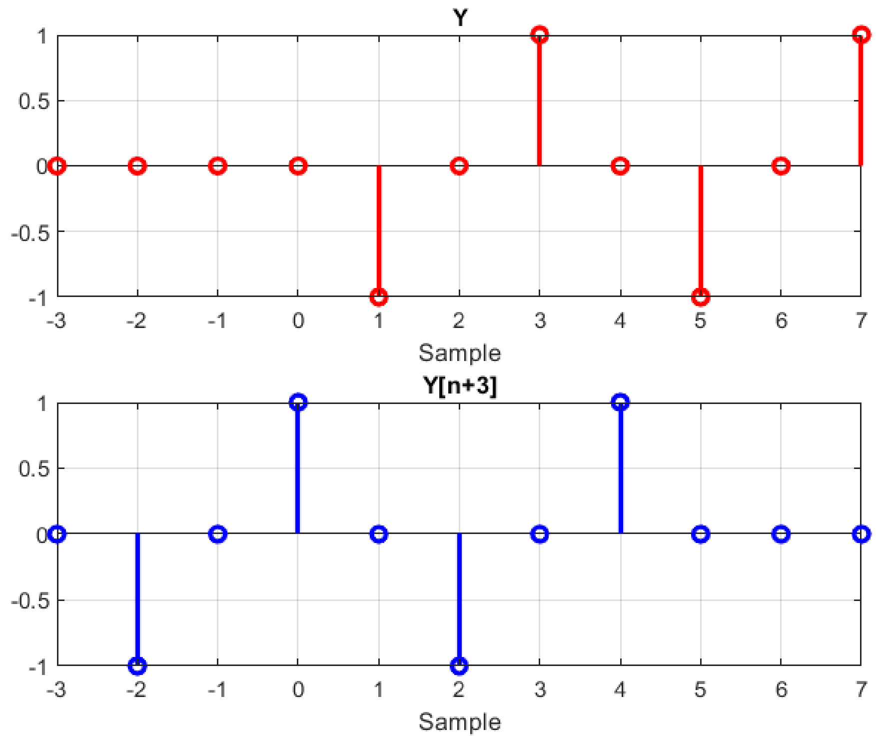


In Class Problem
Time Shifting

- $y[n+3]$ is the waveform x shifted left by 3 samples

$$y[n] = 0,\ 0, 0, 0, -1, 0, 1, 0, -1, 0, 1$$

$$y[n+3] = 0, -1, 0, 1, 0, -1, 0, 1, 0, 0, 0$$



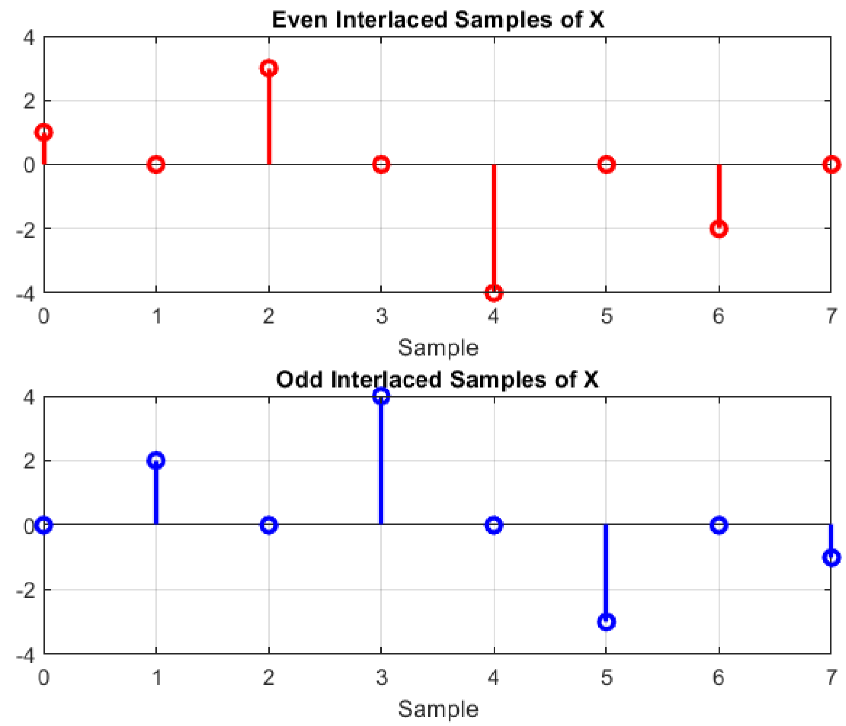
In Class Problem
Interlaced Decomposition

- Assume that the index starts at 0

$$x[n] = 1, 2, 3, 4, -4, -3, -2, -1, 0$$

$$x[n_{even}] = 1, 0, 3, 0, -4, 0, -2, 0$$

$$x[n_{odd}] = 0, 2, 0, 4, 0, -3, 0, -1, 0$$



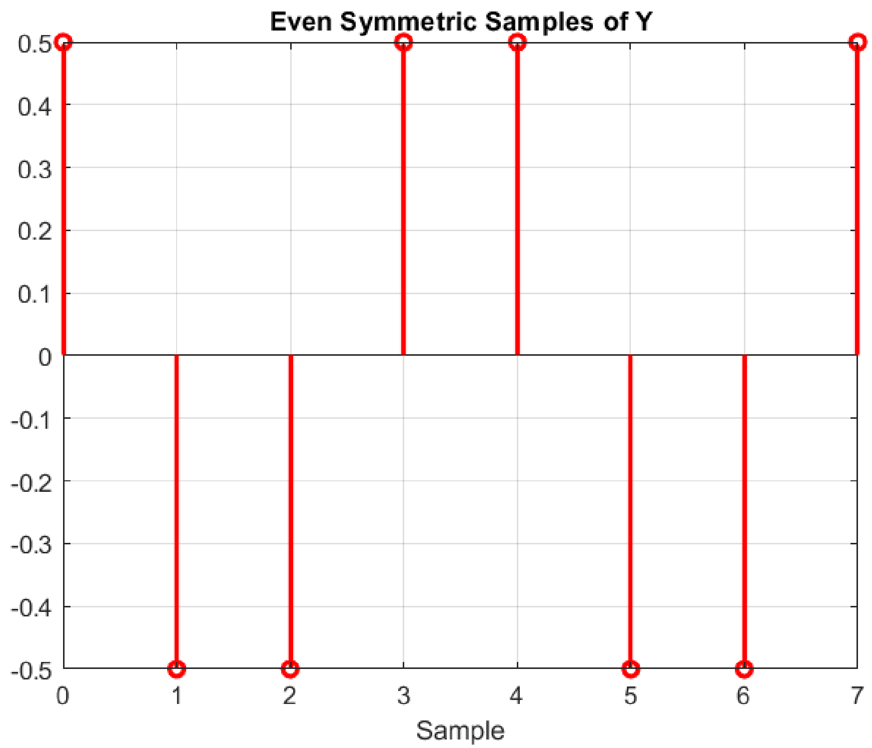
In Class Problem
Even Symmetric Function

$$y_{even} = \frac{y[n] + y[N-1-n]}{2}$$

$$y[n] = \quad 0, -1, 0, 1, 0, -1, 0, 1$$

$$y[N-n] = \quad 1, 0, -1, 0, 1, 0, -1, 0$$

$$y_{even} = \left[\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right]$$



Today's Summary

- For a system to be linear, it must have two properties, **homogeneity** and **additivity**.
- **Homogeneity** - scaling the input signal by a scale factor K causes the output to scale the same factor K.
- **Additivity -** If input signals added together pass through the system without interacting, then the system is additive.

Today' Summary (2)

- A system is **shift invariant** if a shift in the input signal causes an identical shift in the output signal.

- If a system is proven to have both homogeneity and additivity then it can be proven to be linear.

- A system can be linear without being shift invariant.