Digital Signal Processing

Fourier Transform Pairs

Topics

- Duality between time domain and frequency domain
- Investigate the Impulse and SINC functions – time and frequency domain
- Fourier Transform Pairs
  - Gaussian Pulse and Burst
  - Chirp Signals
- Gibbs effect

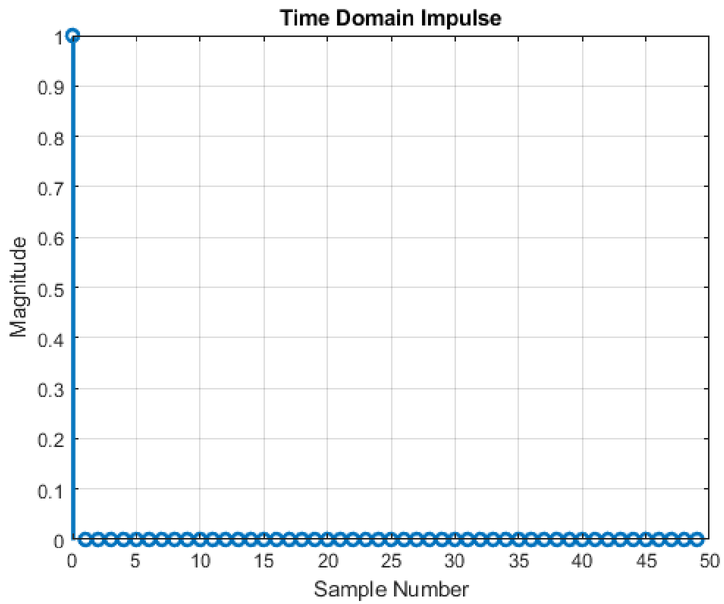
Duality between time and frequency domain

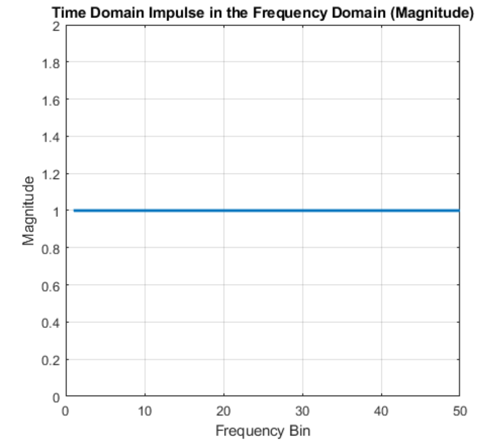
- For every time domain waveform there is a corresponding frequency domain waveform.

- And, for every frequency domain waveform there is a corresponding time domain waveform.

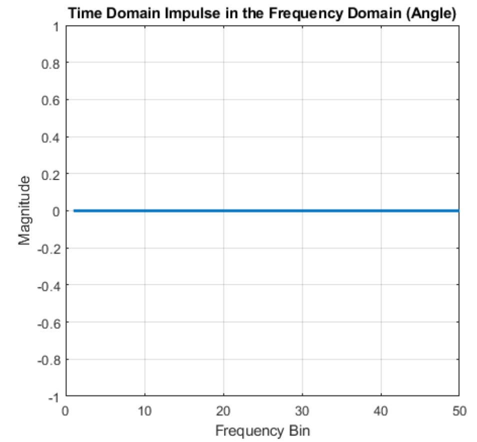
- We saw this in creating the ideal LPF with the SINC impulse response

Some Example Time/Frequency Domain Pairs – Delta Function





Magnitude

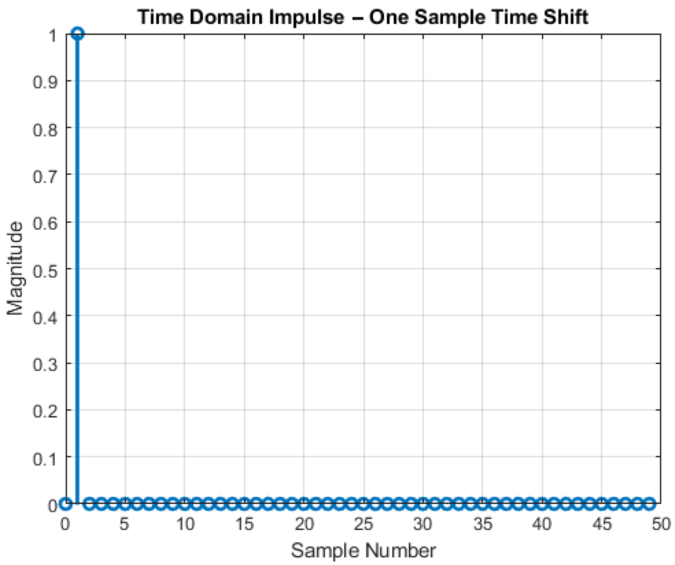


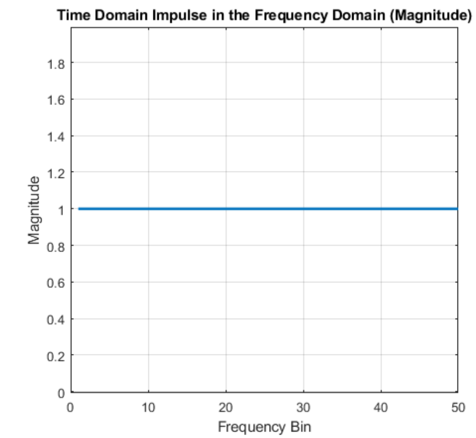
Phase

An impulse in the time domain results in a frequency response that is flat over all frequencies.

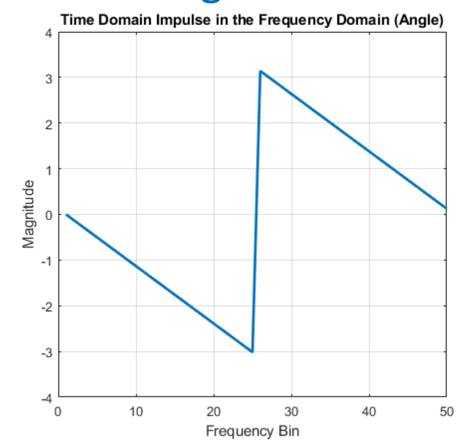
Phase response is zero for all frequencies as well

Time Shift = Linear phase Shift
One Sample





Magnitude

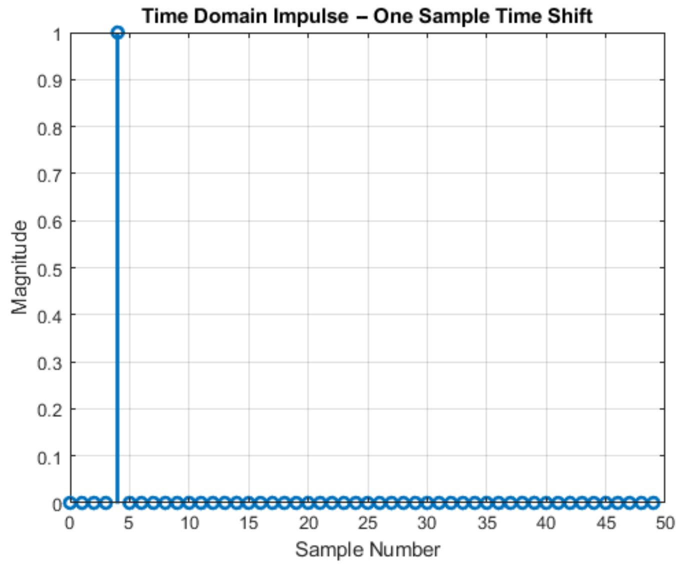


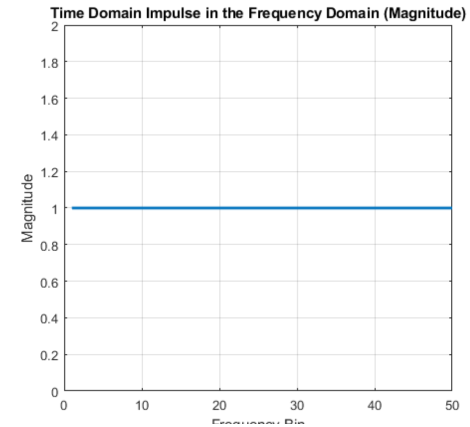
Phase

An impulse shifted in the time domain results in a frequency response that is still flat over all frequencies.

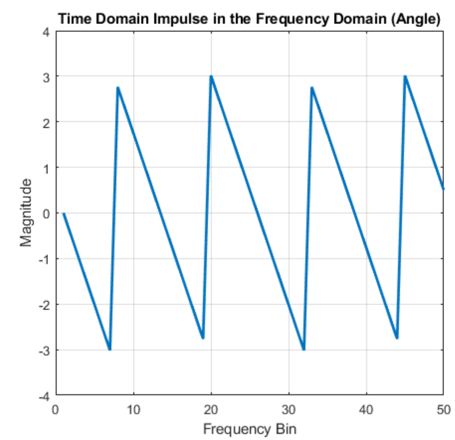
Phase response is linearly changing over all frequencies

Time Shift = Linear phase Shift
Five Samples





Magnitude

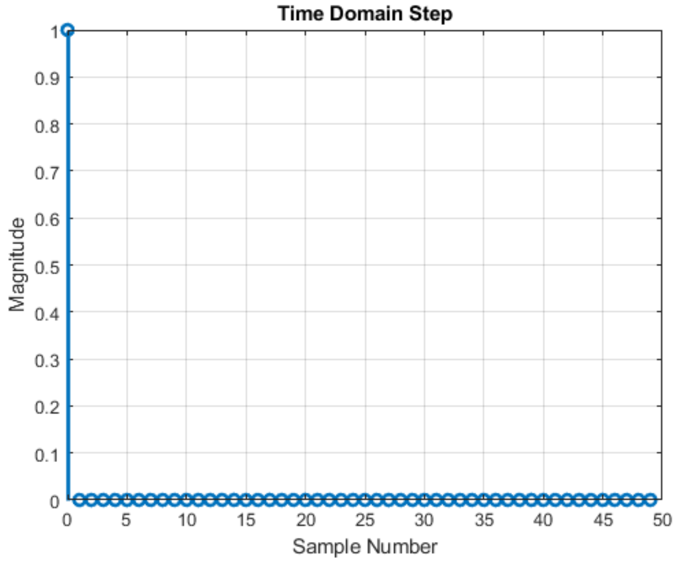


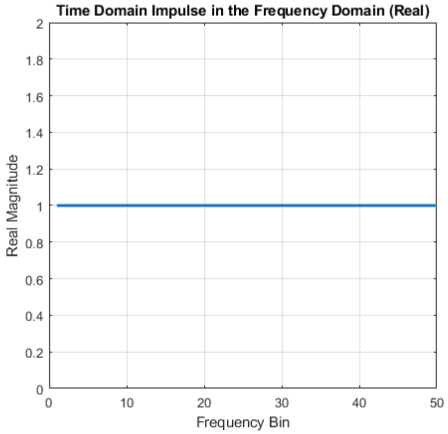
Phase

An impulse shifted more in the time domain the frequency response is still flat over all frequencies.

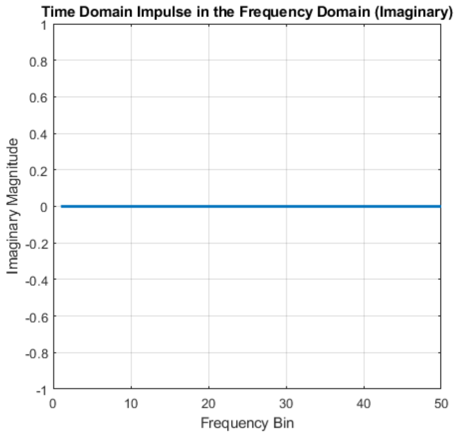
Phase response is linearly changes more rapidly over frequency

Freq Domain Delta Function DFT Real and Imaginary Terms





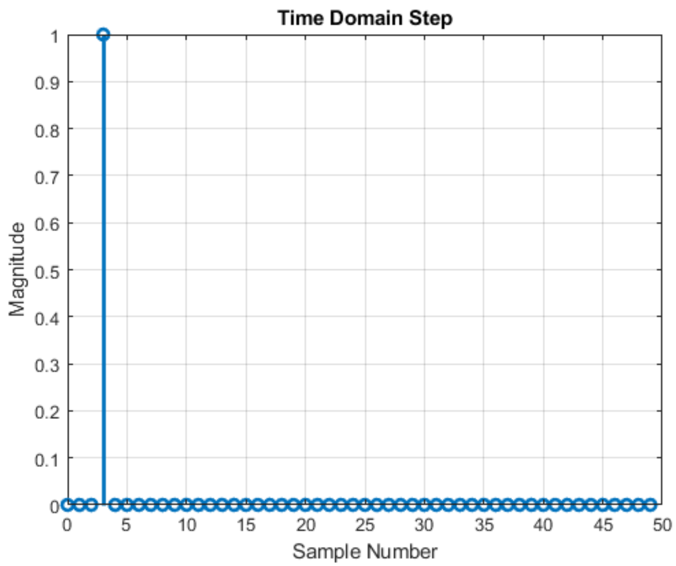
Real

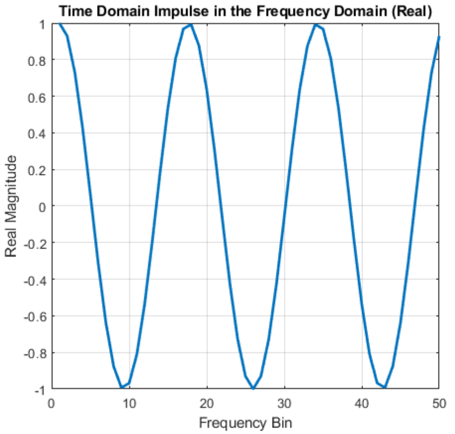


For an impulse at zero the real portion is 1 and the imaginary term is 0 in the frequency domain

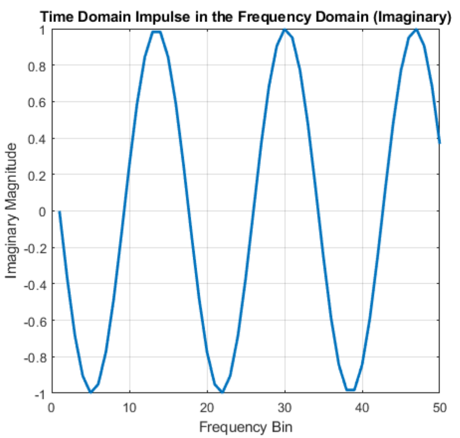
Imaginary

Freq Domain Delta Function DFT Real and Imaginary Terms





Real



Imaginary

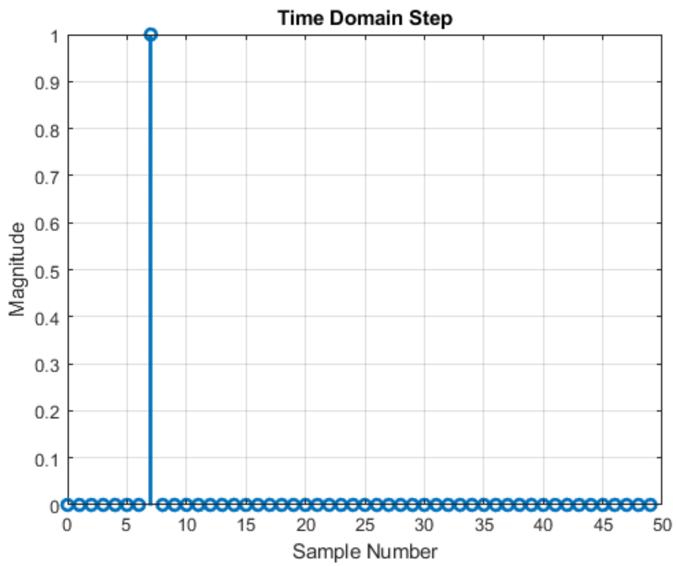
For a time shifted impulse the real and imaginary parts of the frequency domain are sinusoidal

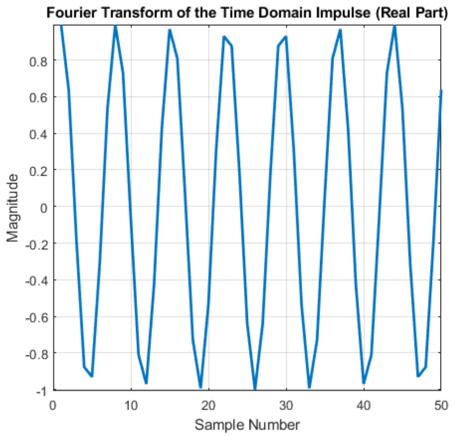
The phase shift corresponds to a complex exponential

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

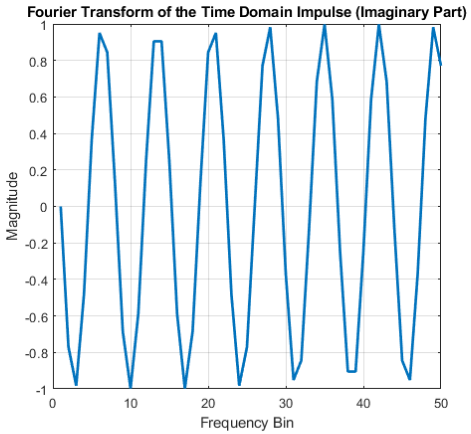
Impulse at 3rd sample – 3 cycles in the frequency domain

Time shifted impulse = Sinusoidal Real and Imaginary Part





Real



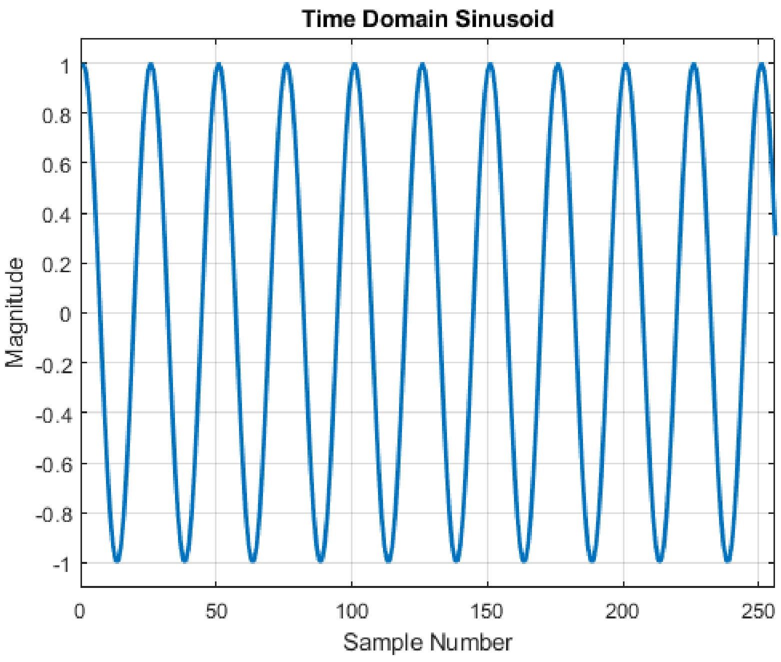
Additional time shift increases the number of cycles in the frequency domain

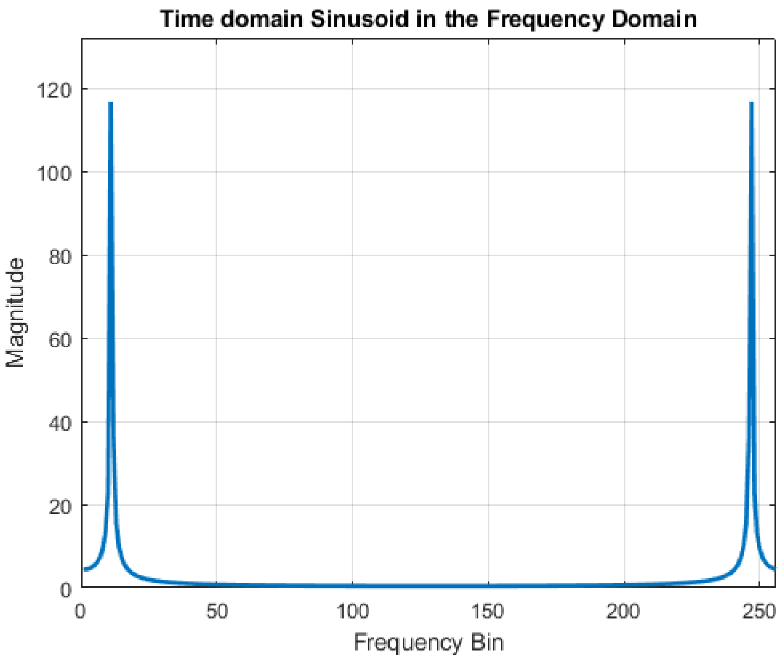
Impulse at 7 samples – 7 cycles in the frequency domain

Imaginary

Sinusoid in the Time Domain

- A Sinusoid in the time domain is an impulse in the frequency domain



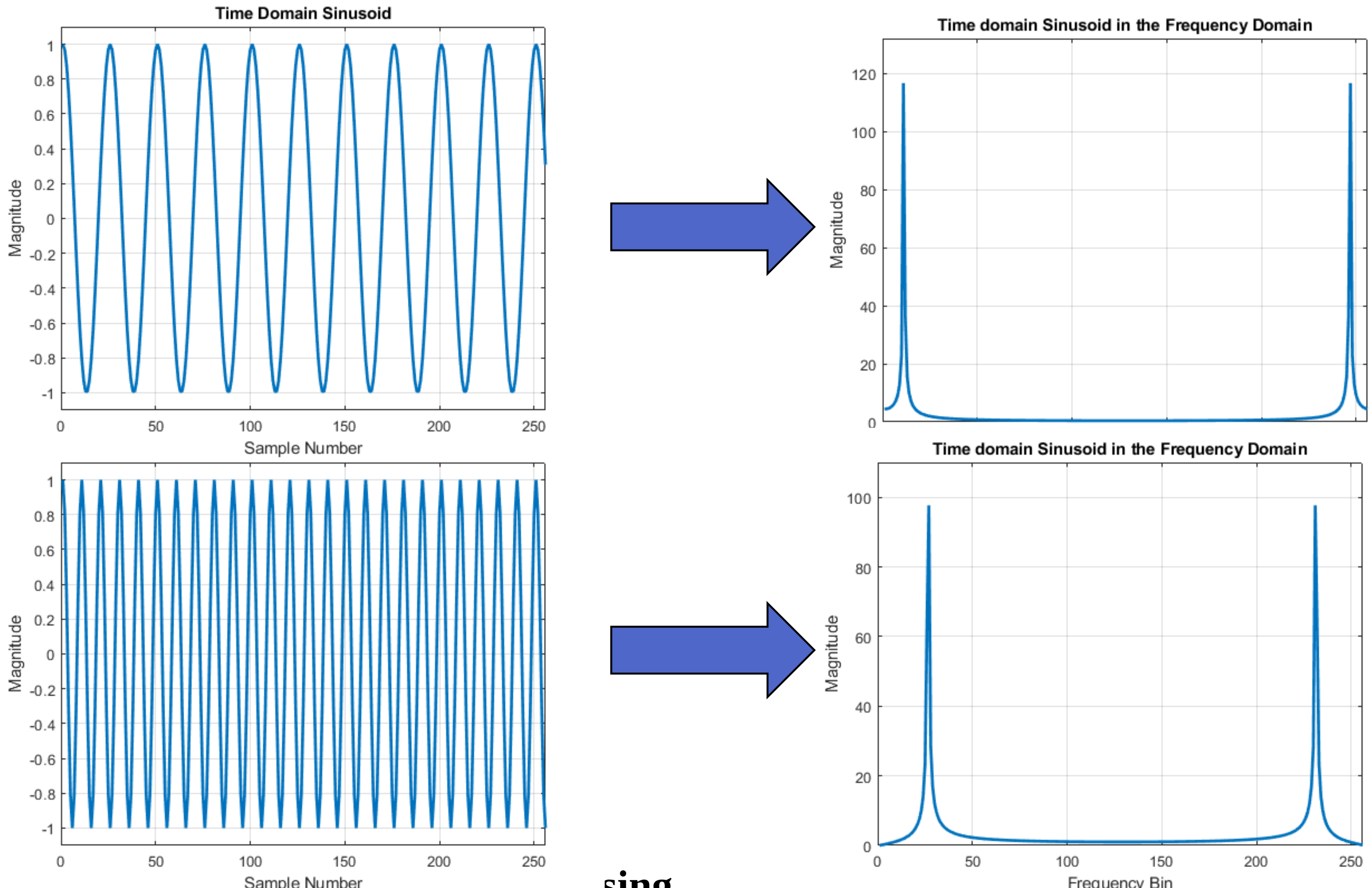


Time Domain

Frequency Domain

Sinusoid in the Time Domain

- As the frequency increase the impulse in the frequency domain shifts



Time and Frequency Domain Duality

- Each sample in the time domain corresponds to a sinusoid in the frequency domain

Time/Frequency Domain Duality

- Each sample in the frequency domain corresponds to a sinusoid in the time domain.

Time and Frequency Domain Duality

- An impulse at sample #4 creates a four cycles of cosine and four cycles of negative sine in the frequency domain.
- We can use this property to synthesize the frequency domain from an impulse decomposition in the time domain
- Likewise we can use the property to synthesize the time domain signal from the components in the frequency domain

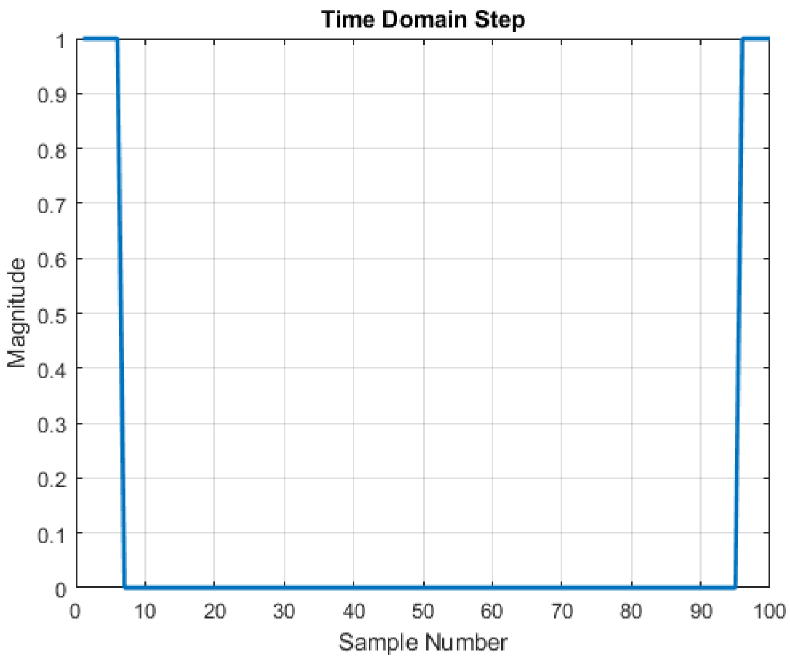
SINC Function

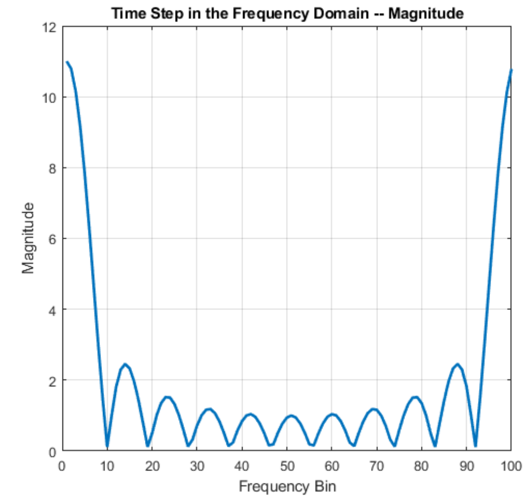
- A useful Fourier Transform pair is that of a rectangular pulse and the SINC function
- SINC function defined as

$$SINC(x) = \frac{SIN(\pi x)}{\pi x}$$

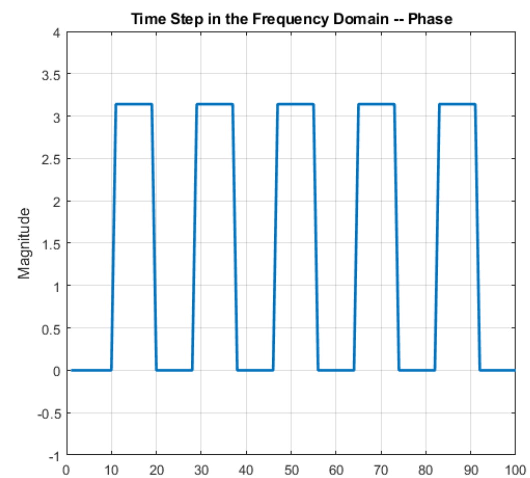
- Where the rectangular pulse is centered on sample zero.

Step in Time Domain
SINC in Frequency Domain





Magnitude



Phase

Time domain pulse is centered
on 0 repeats every 128 samples

SINC Function the frequency domain

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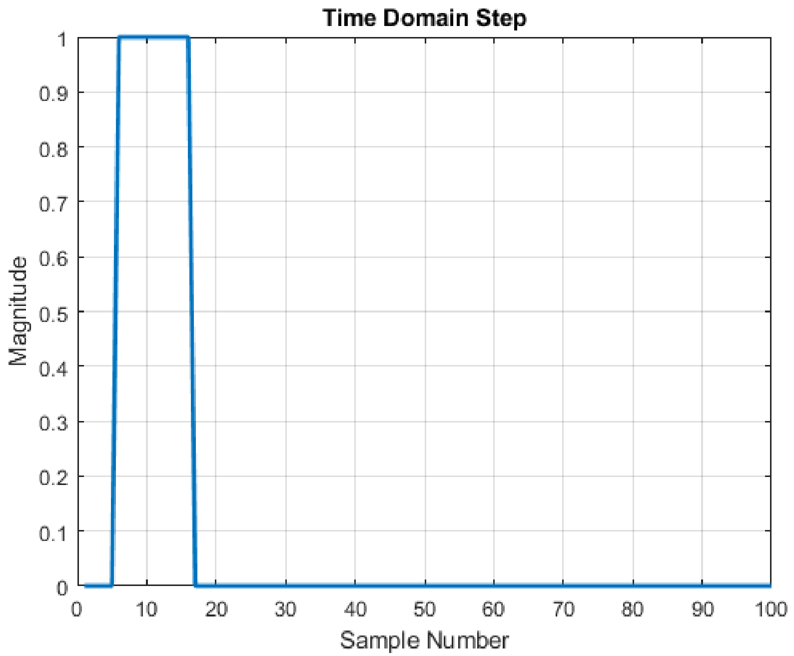
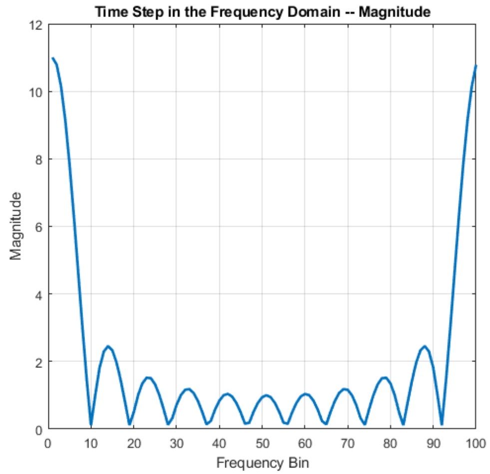
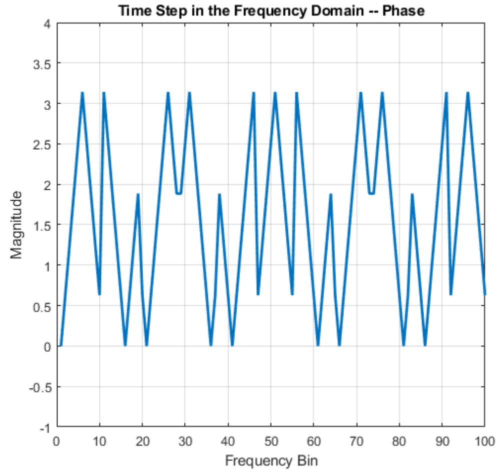
Step in Time Domain
SINC in Frequency Domain





Magnitude

Time domain pulse shifted in the time domain

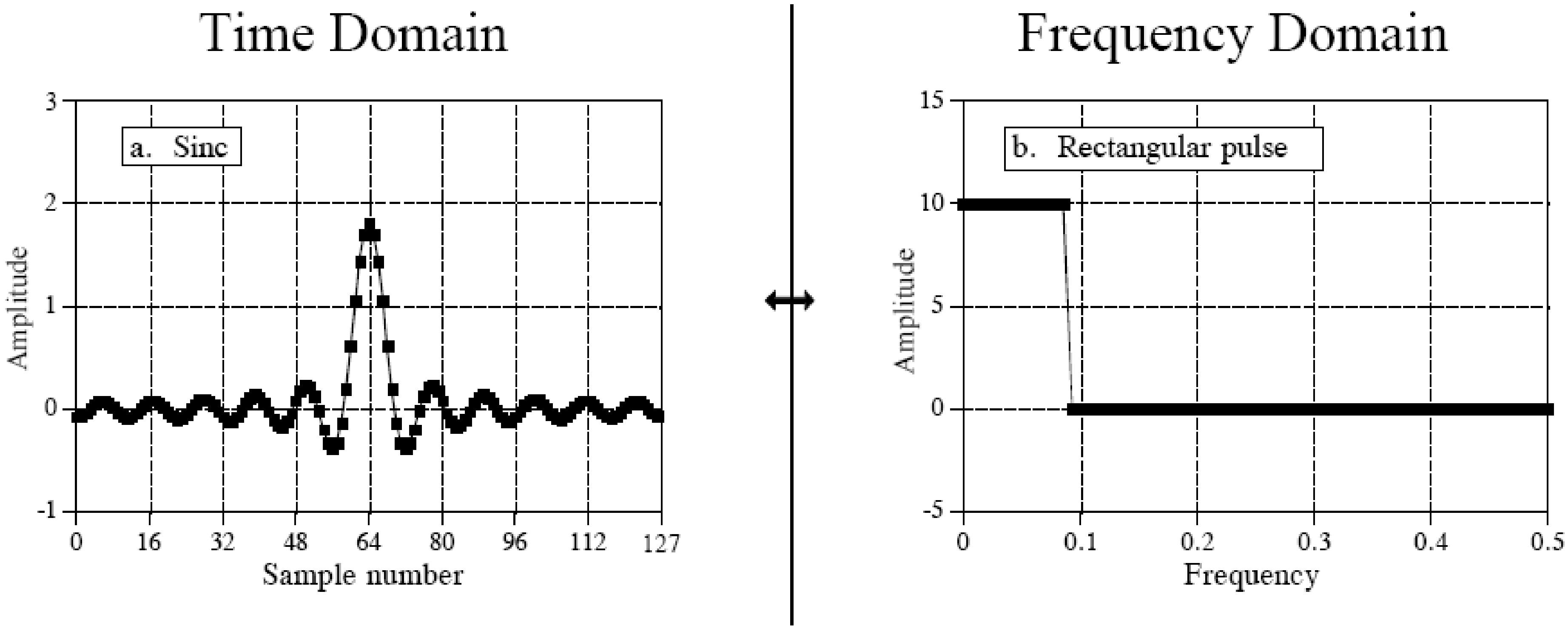


Magnitude is the same. Linear phase shift

Phase

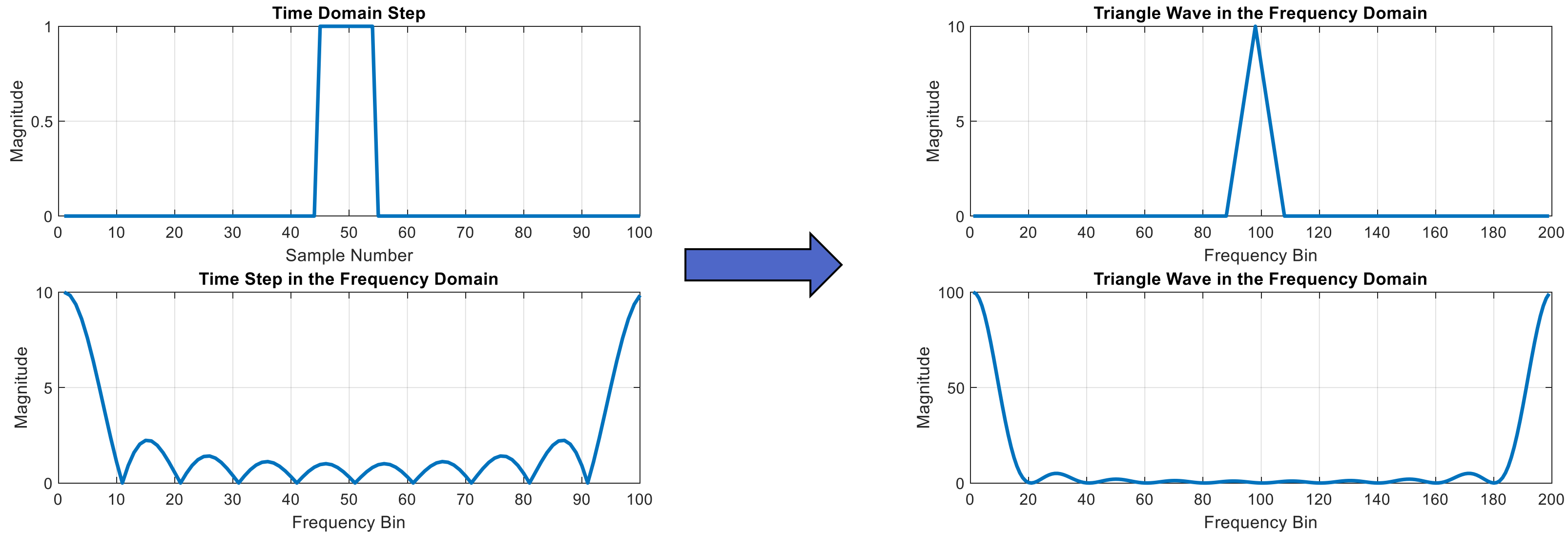
Fourier Transform Pairs

- Ideal low pass filter in frequency domain comes from a SINC impulse response in the time domain -- unrealizable



Triangle Wave

- Convolving two pulses in the time domain results in a triangle wave



- This results in a multiplication of the two SINC functions in the frequency domain $SINC^2$

Convolution/Multiplication Duality

- Convolution in the time domain results in multiplication in the frequency domain

$\Updownarrow$ Time/Frequency Domain Duality

- Convolution in the frequency domain results in multiplication in the time domain

Gaussian Wave

- Gaussian wave in time domain is the Gaussian in frequency domain. (Note negative frequencies make for symmetry in frequency domain)

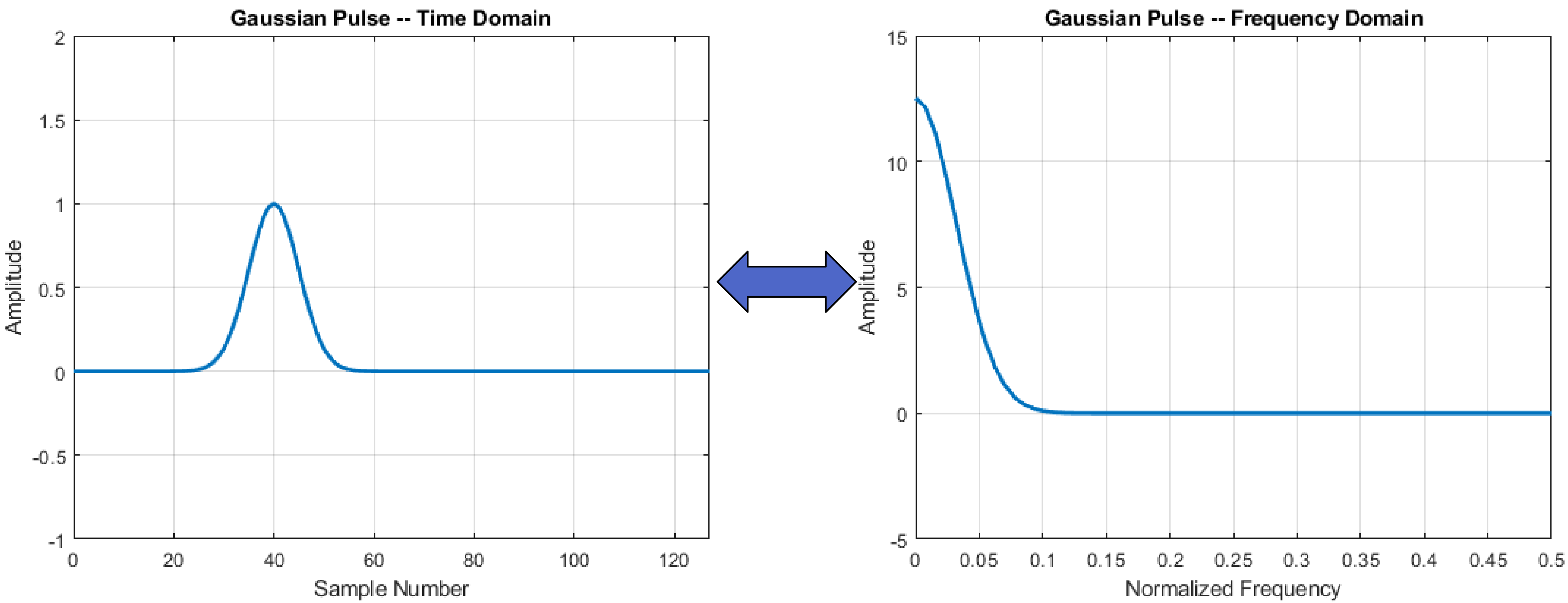
- The Gaussian pulse can be described by the equation

$$y(x) = ae^{\frac{-(x-b)^2}{2c^2}}$$

- The normal or Gaussian probability density function follows this shape

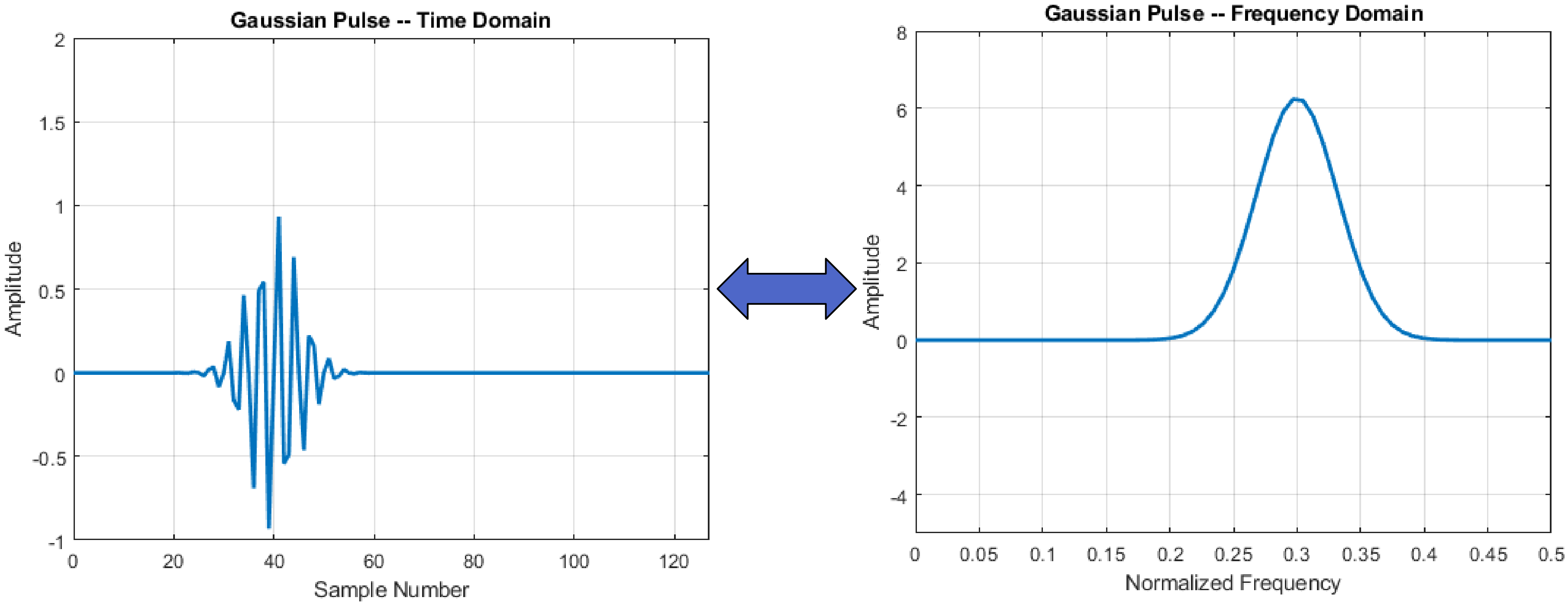
Gaussian Wave

- Gaussian wave in time domain is the Gaussian in frequency domain. (Note negative frequencies make for symmetry in frequency domain)



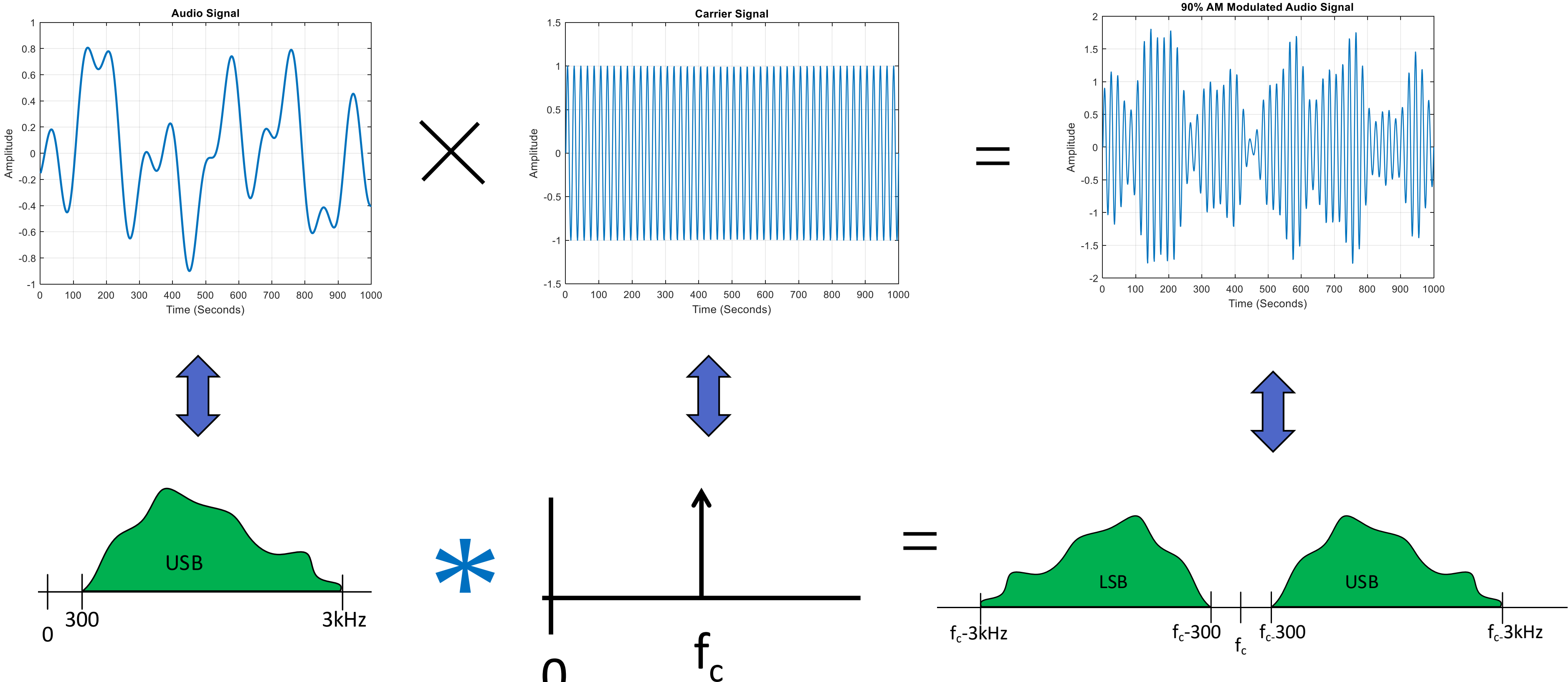
Gaussian Burst

- Gaussian Burst (gaussian times sine wave) in time domain is the Gaussian time shifted (convolved with delta function) in frequency



Amplitude Modulation Example (DSB-SC)

Multiplying by a sinewave in the time domain causes a frequency shift in the frequency domain



Negative Frequencies

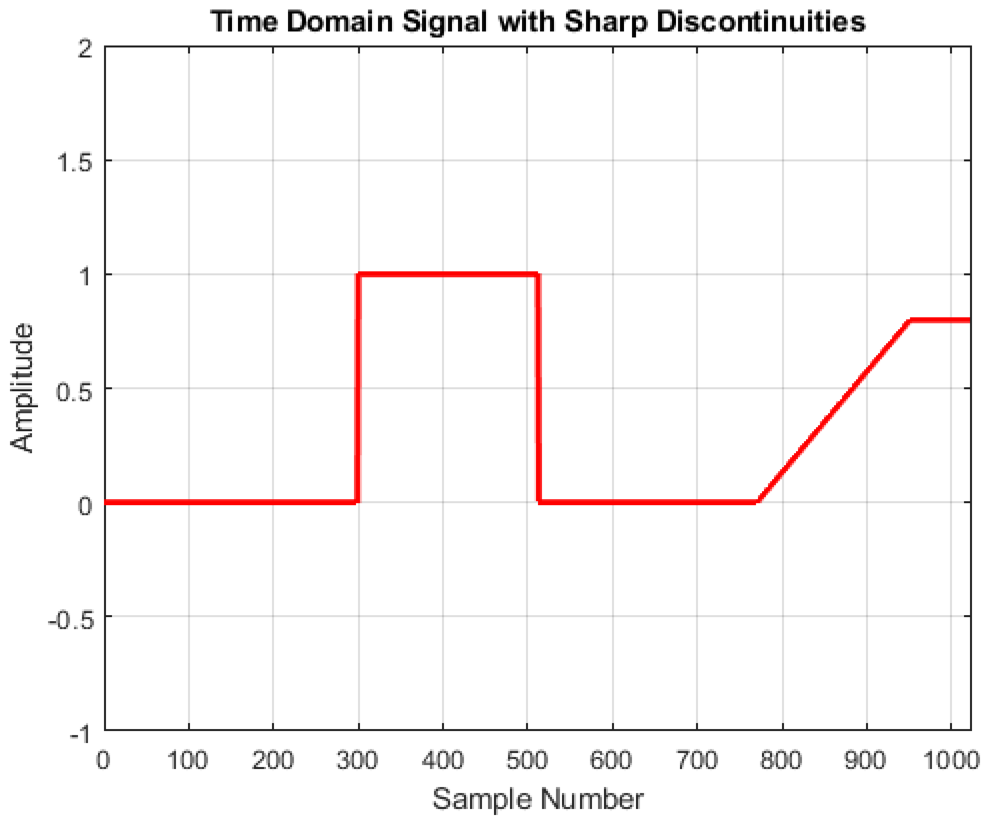
Positive Frequencies

Gibbs Effect

- A time domain signal that has a step discontinuity will have overshoot and ripple near the discontinuities when it is synthesized from sinusoids.
- Known as the Gibbs effect.
- As number of sinusoids used in synthesis increases, the energy of the overshoot with ringing of the sinusoidal synthesis goes to zero.
- Becomes a more accurate representation of the response

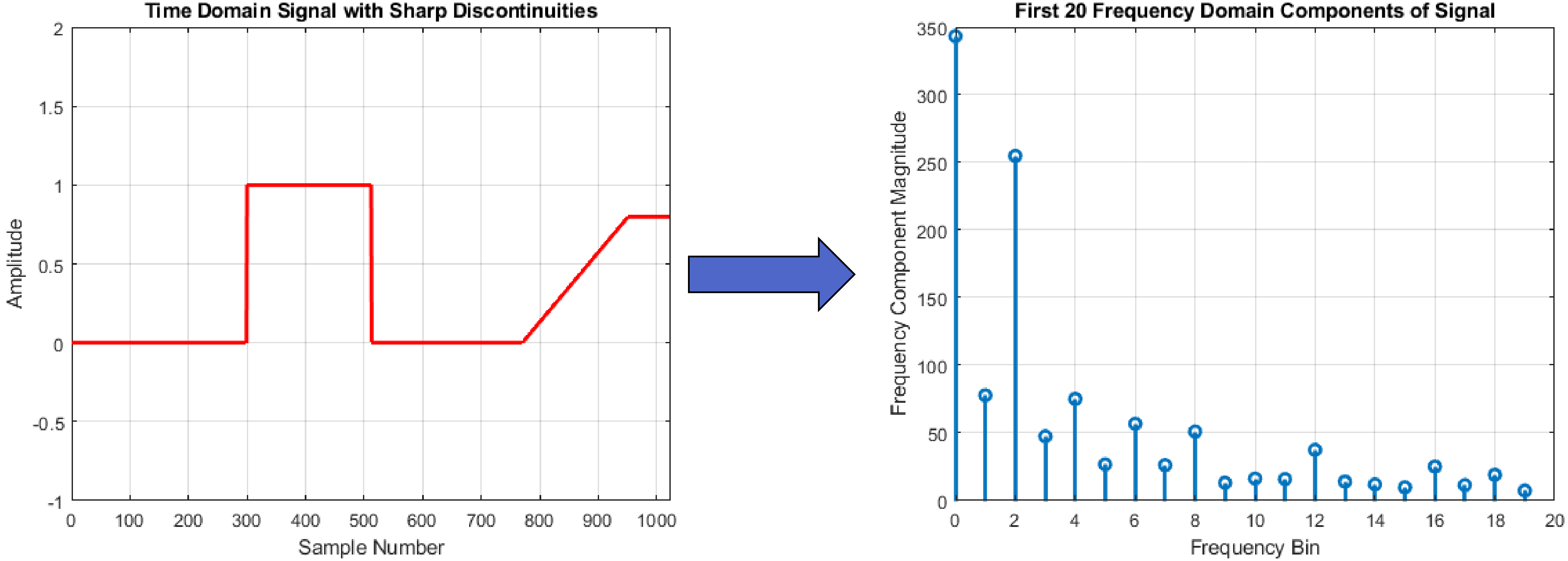
Gibbs Effect Example

- Start with a signal with sharp discontinuities



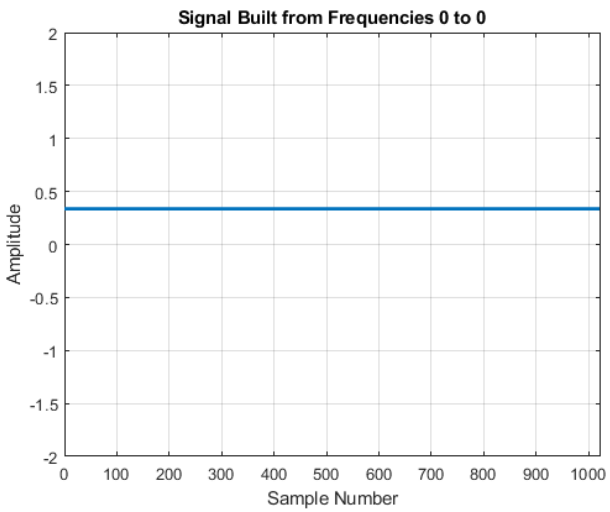
Gibbs Effect Example

- Look at the first 20 frequency domain components by computing the FFT

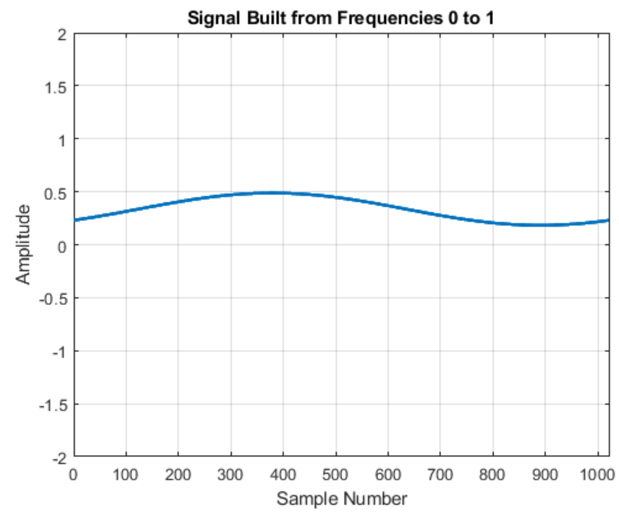


Build the Signal from Sinusoidal Frequencies

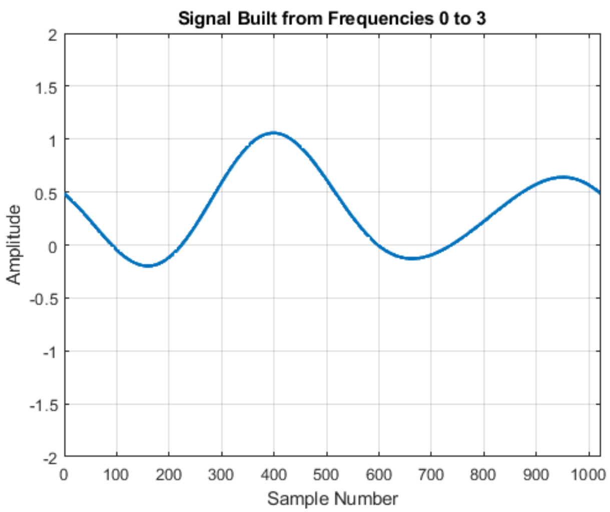
0



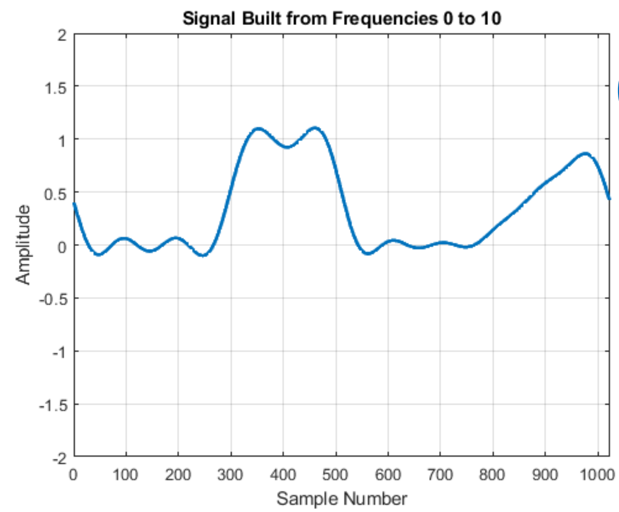
0-1



0-3

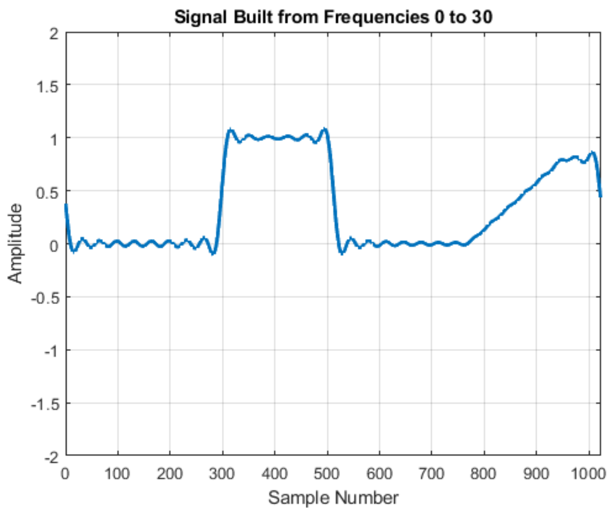


0-10

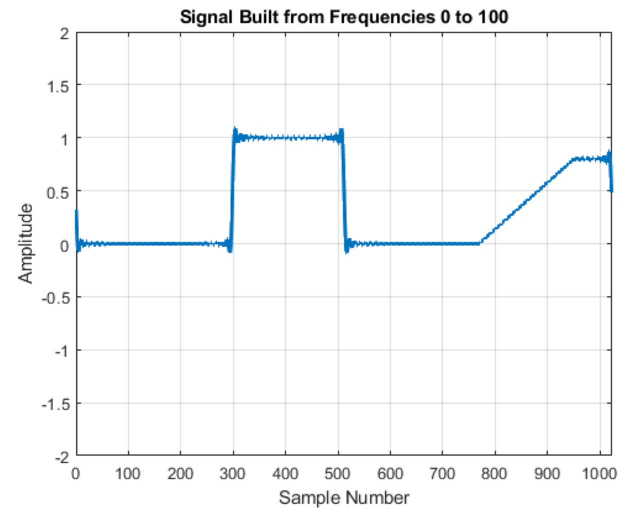


Build a Signal from Sinusoidal Frequencies

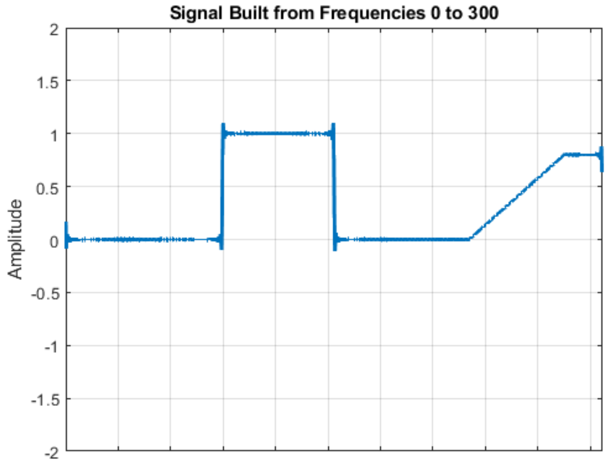
0-30



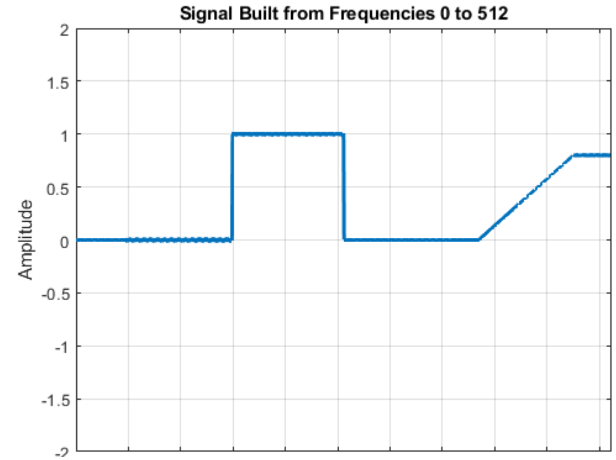
0-100



0-300



0-512



Adding more frequencies increases the detail and definition of discontinuities

Impact of Gibbs Effect

- A low pass filter will truncate higher frequency content of a signal.
- This can result in overshoot and ringing in the time domain.

↕ Time/Frequency Domain Duality

- Likewise truncating the time domain signal will cause distortion of the edges in the frequency domain.
- This will affect filter design (as in Windowed SINC filter)

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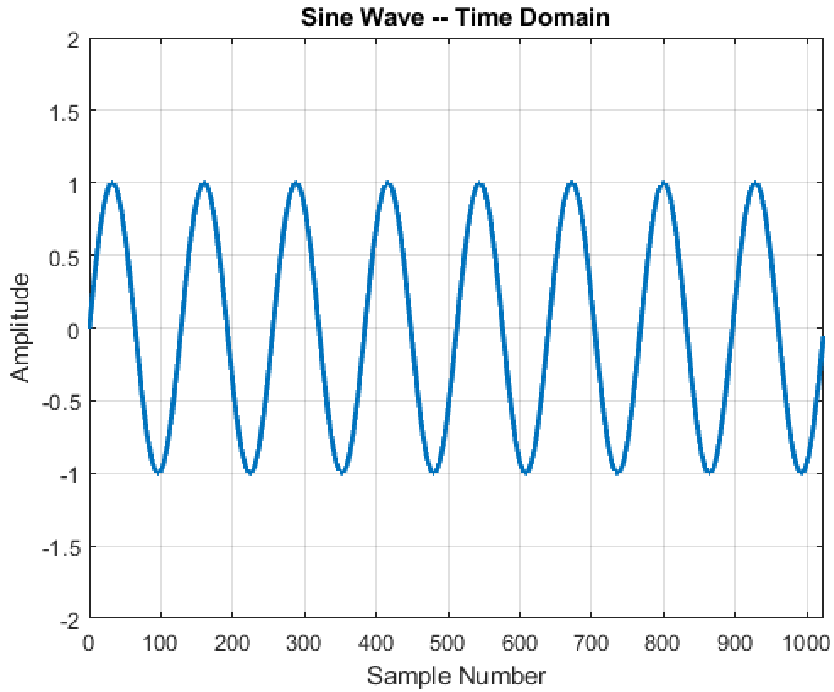
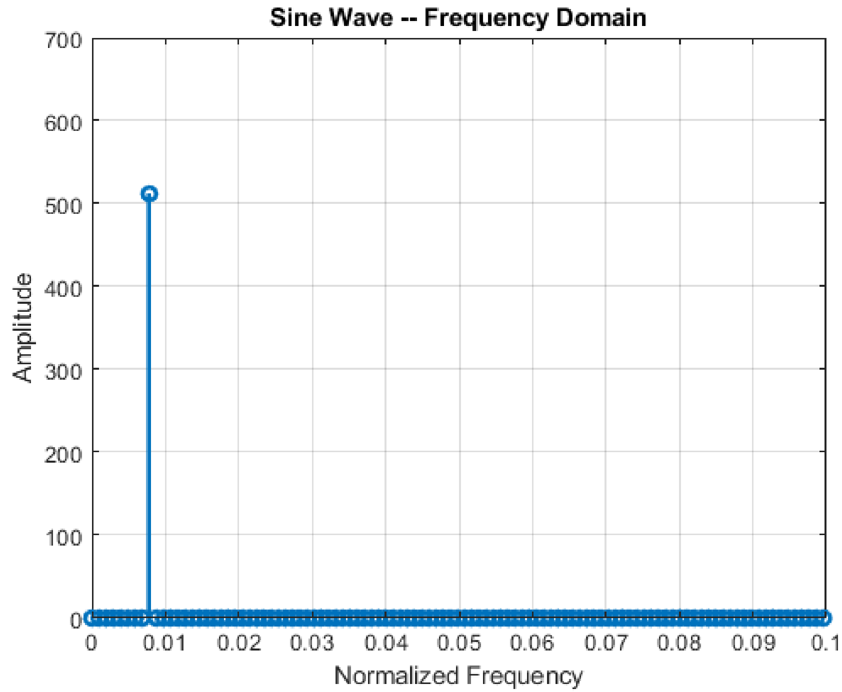
Harmonics

- If a signal is periodic with frequency f, then frequencies composing the signal are integer multiples of f
  - That is f, 2f, 3f, 4f, etc.
- These frequencies are called harmonics.
- The first frequency f, is called the fundamental
- 2f is the second harmonic, etc.

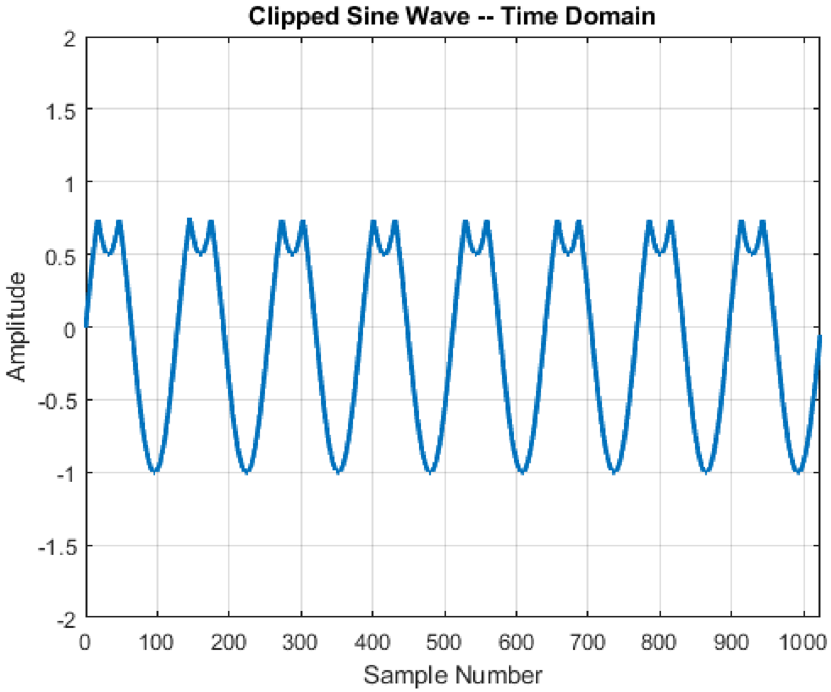
Harmonics

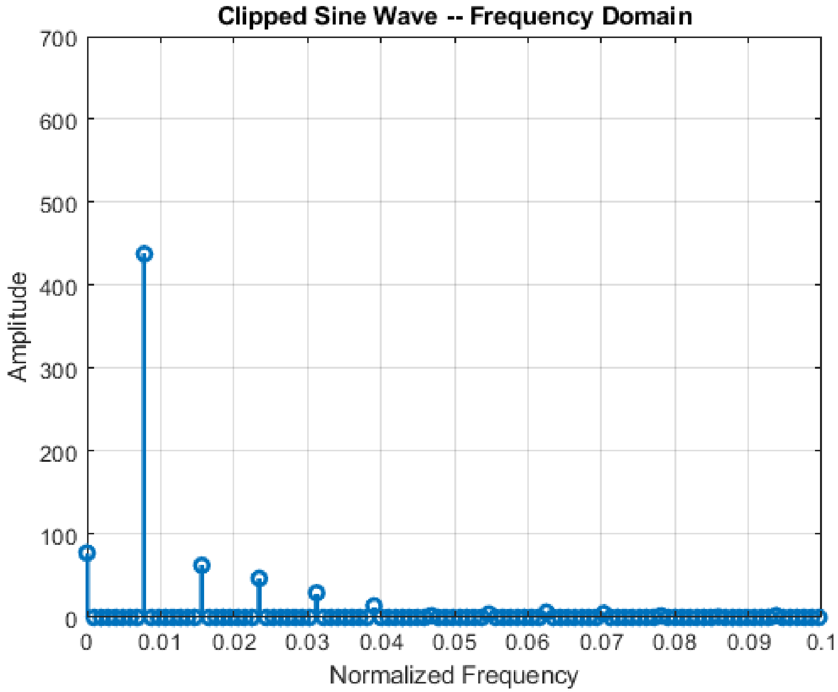




A pure sine wave is made up of a single frequency at the fundamental frequency

Harmonics – Asymmetrical Distortion

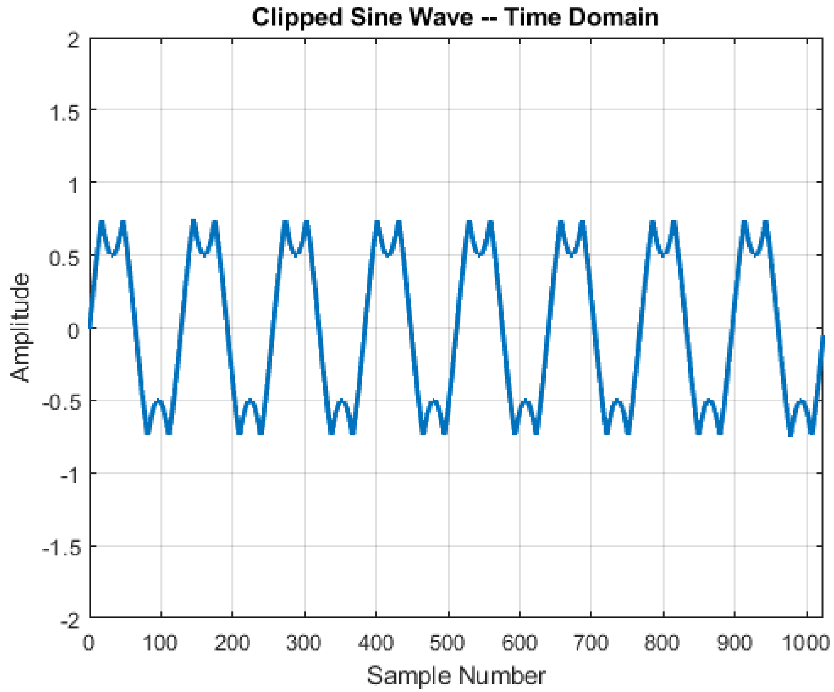


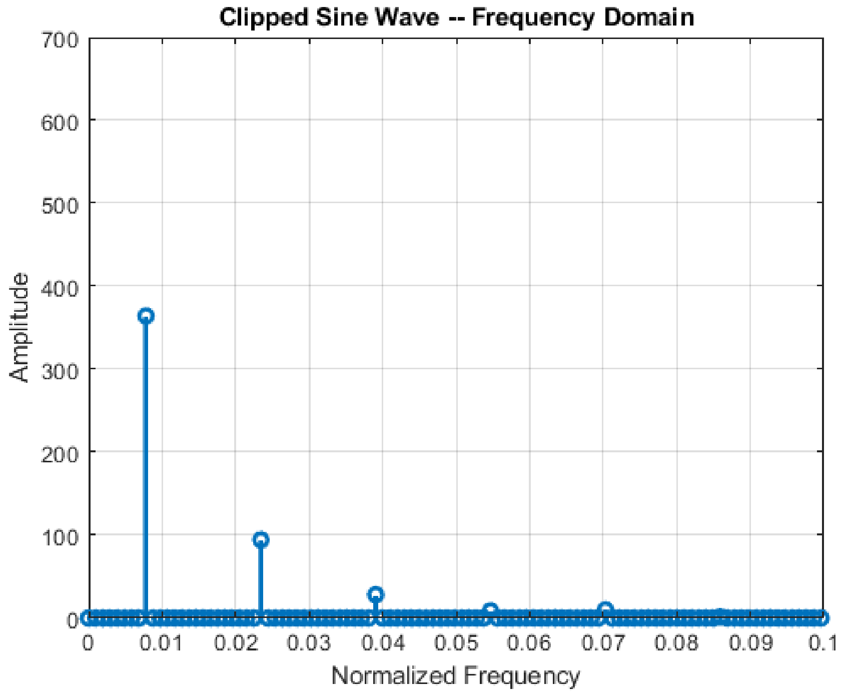


A periodic, but non-sinusoidal signal will have harmonics in addition to the fundamental frequency

Distortion will create harmonics of a signal

Harmonics – Asymmetrical Distortion

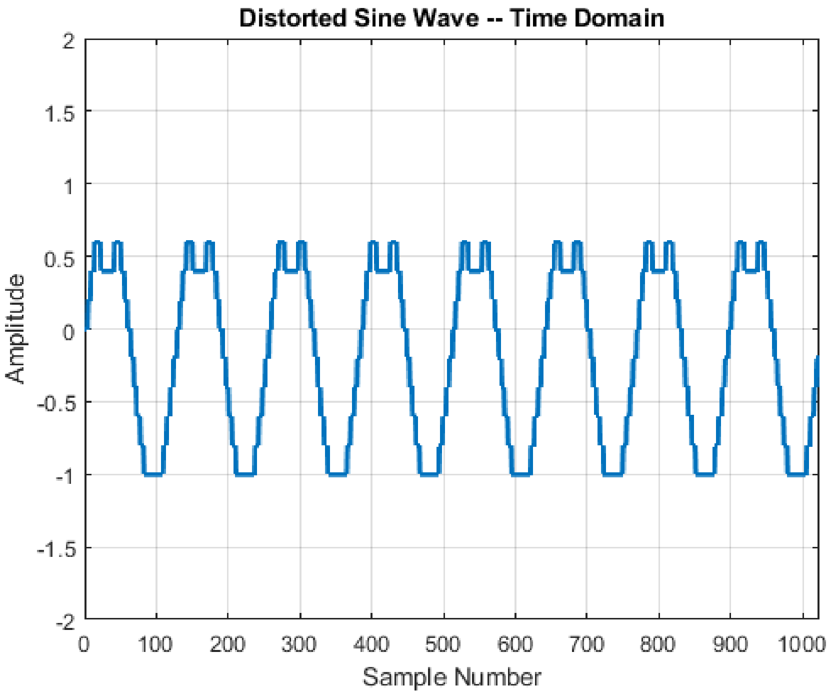


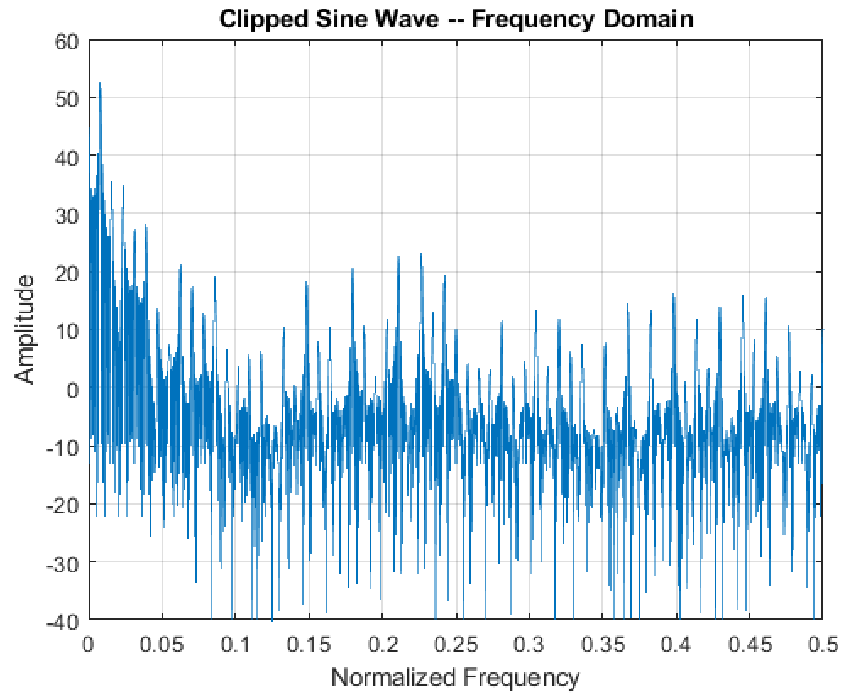


If the distortion is symmetric about the horizontal axis, only odd harmonics appear

Harmonics and Aliasing

Distortion in the discrete domain that causes high frequency harmonics (> Fs/2) will alias into the range from 0 to Fs/2





Decibel Scale

Radar and Sonar

- In echo location systems (like radar) a short RF pulse is transmitted
- A receiver looks for the reflected signal from an object
- The time difference between the transmission and the reception can be used to calculate the distance to the object.

Radar and Sonar

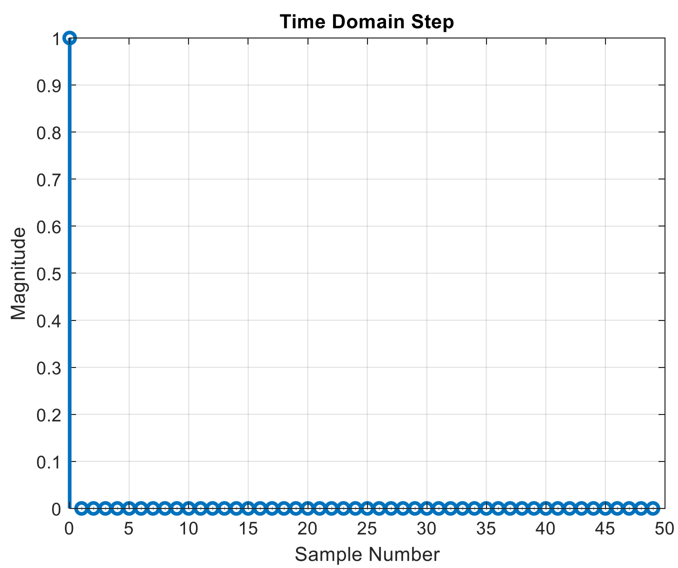
- The narrower (shorter) the transmission the better resolution that is possible
  - Impulse-like signal

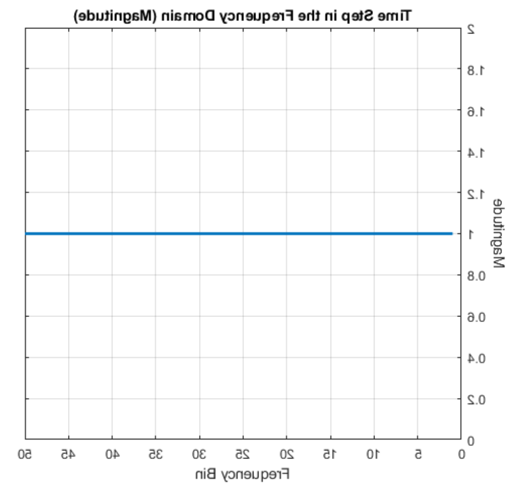
- The greater the signal power the longer the range of detection will be

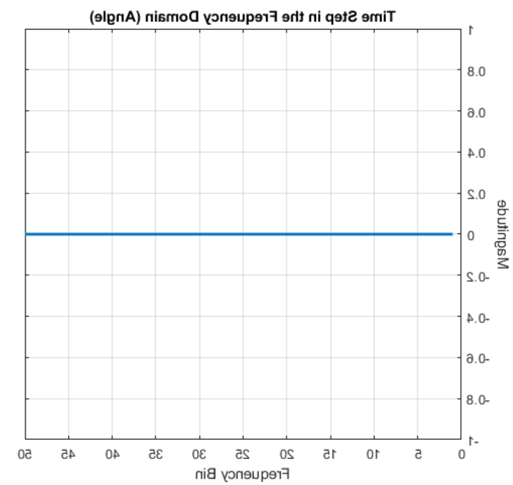
- It is difficult to create a very short impulse-like signal with very high power

Chirp Signals

- An impulse has a flat frequency response and zero or linear phase shift







Chirp Signals

- An ideal chirp signal has a flat frequency response but a parabolic phase response

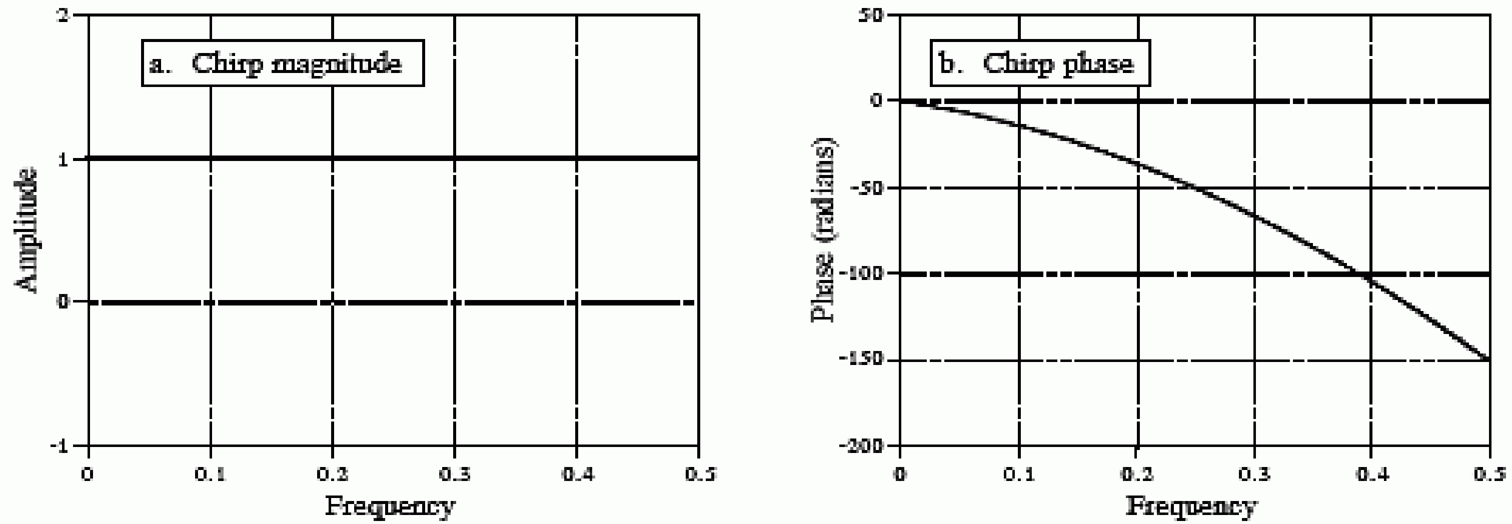
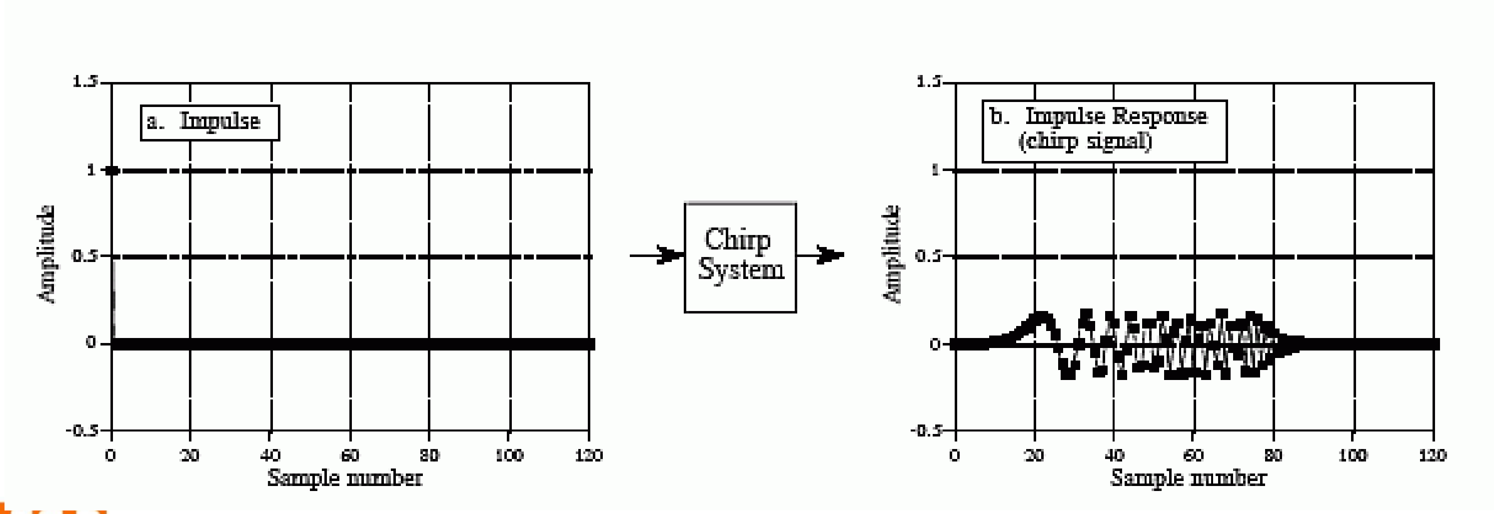


FIGURE 11-9
Frequency response of the chirp system. The magnitude is a constant, while the phase is a parabola.

Chirp Signals

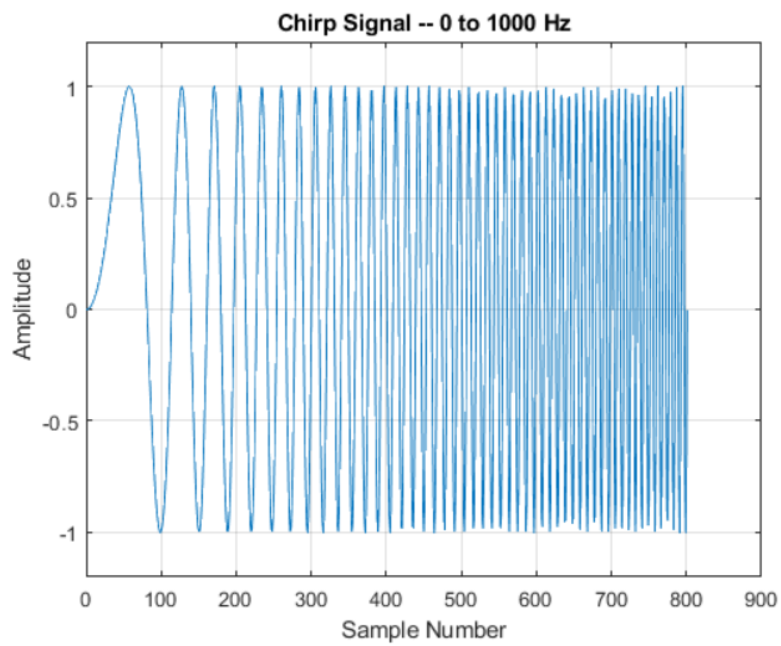
- A Chirp signal is created by starting with a low power impulse (easier to make at low power).

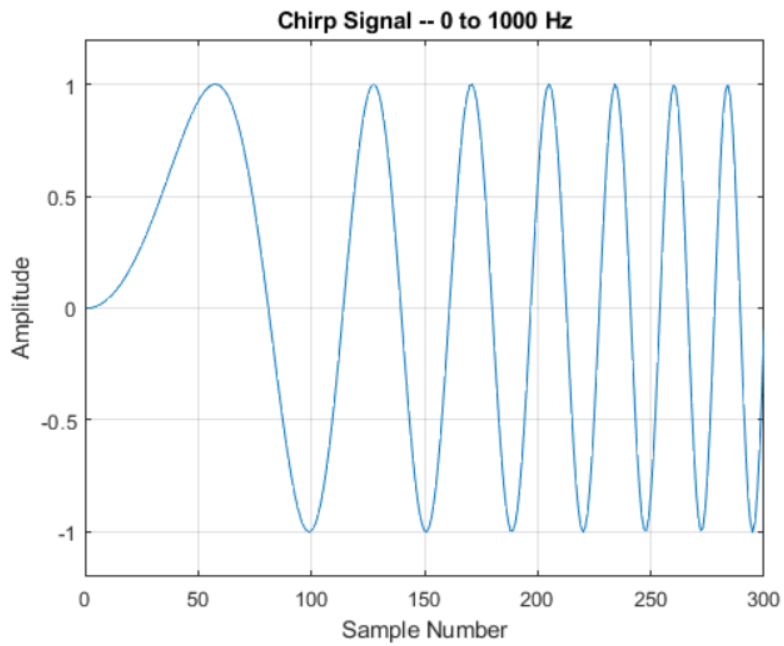
- The impulse is passed through a filter with the impulse response as shown



Chirp Signals

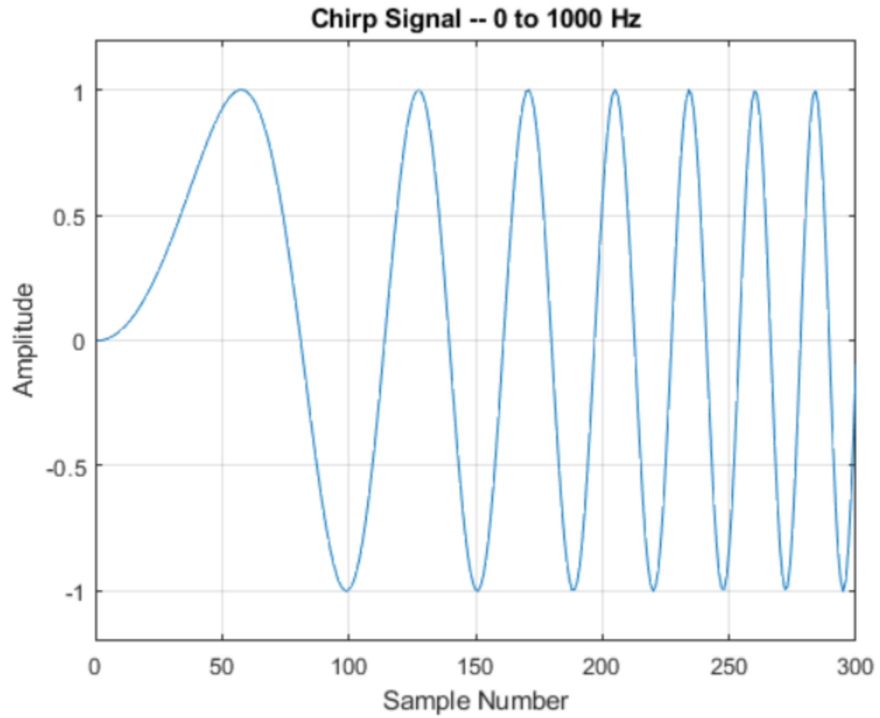
- The chirp is a swept sinusoid
- Freq starts at 0 Hz and linearly sweeps the frequency to a maximum frequency





Chirp Signals

- The signal can more easily be amplified and transmitted
  - Finite amplitude and time signal



Chirp Reception

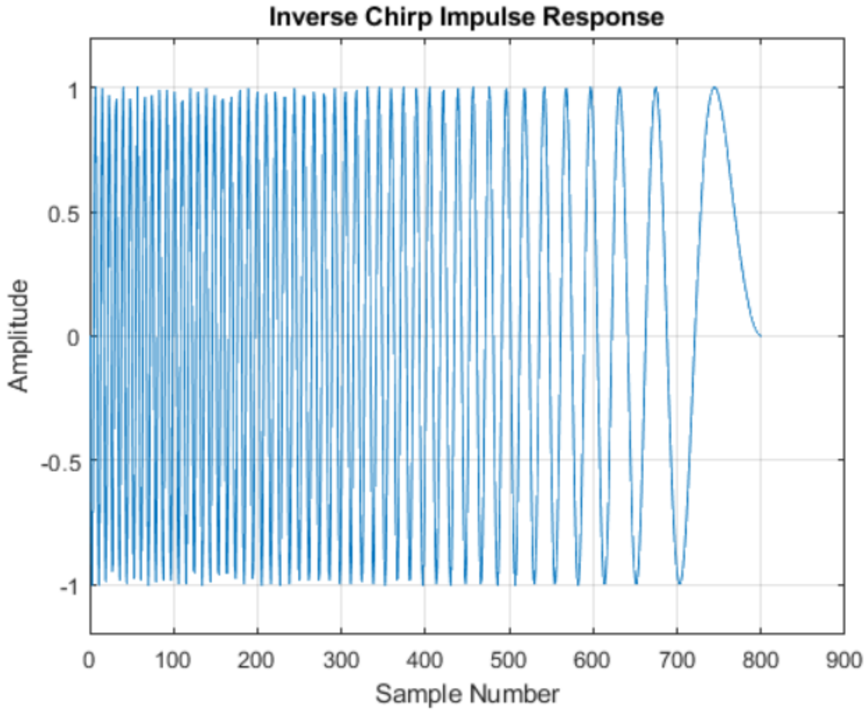
- The signal is reflected from an object and received

- It is then run through an inverse filter (anti-chirp) and the impulse is recovered

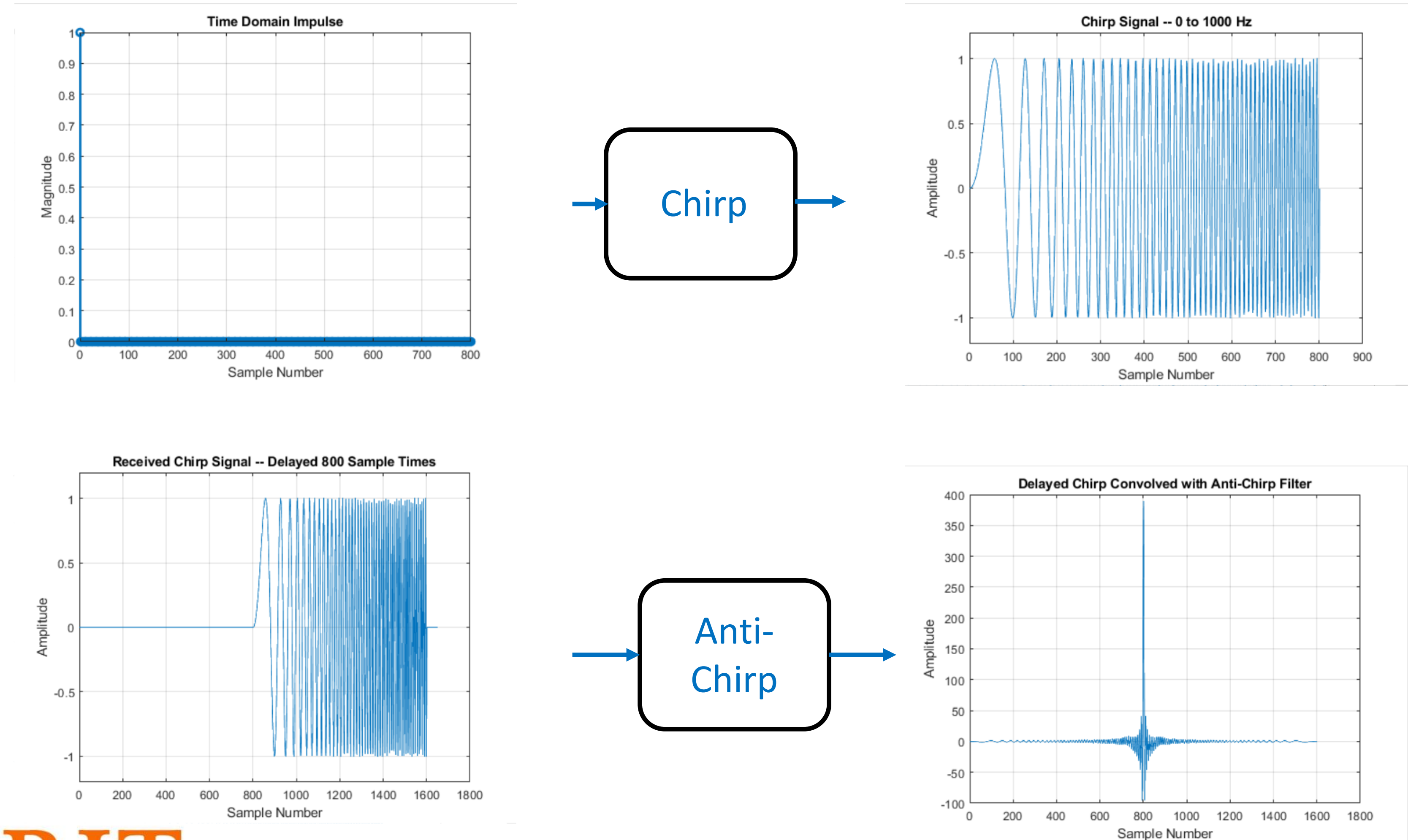
- The time delay can be measured determining the distance to the object

Anti-Chirp Filter

- The anti-chirp filter has a impulse response that is reversed in time from the chirp impulse response – Matched Filter



Chirp System



Summary

- Discussed duality between time domain and frequency domain
- Looked at the Impulse and SINC functions – time and frequency domain
- Fourier Transform Pairs
  - Gaussian Pulse and Burst
  - Chirp Signals
- Gibbs effect

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