Properties of the Discrete Fourier Transform

Today's Topics

- Time domain convolution performed in the frequency domain
- Circular Convolution
- Time domain multiplication and the frequency domain
- Linear properties of the Fourier Transform
- Time shift and Time compression and their impact on the frequency domain

Time Domain/Freq Domain Duality

- Convolution in the time domain is equivalent to multiplication in the frequency domain

$$x[n] * y[n] \leftrightarrow X[k] \times Y[k]$$

- Multiplication in the time domain is equivalent to convolution in the frequency domain

$$x[n] \times y[n] \leftrightarrow X[k] * Y[k]$$

Convolution Example

- Let's say that I have an input signal $x[n]$ in the time domain with length M
- I want to filter the signal with an impulse response $h[n]$ with length N

- To find the output $y[n]$ I would convolve the input x with the filter impulse response h

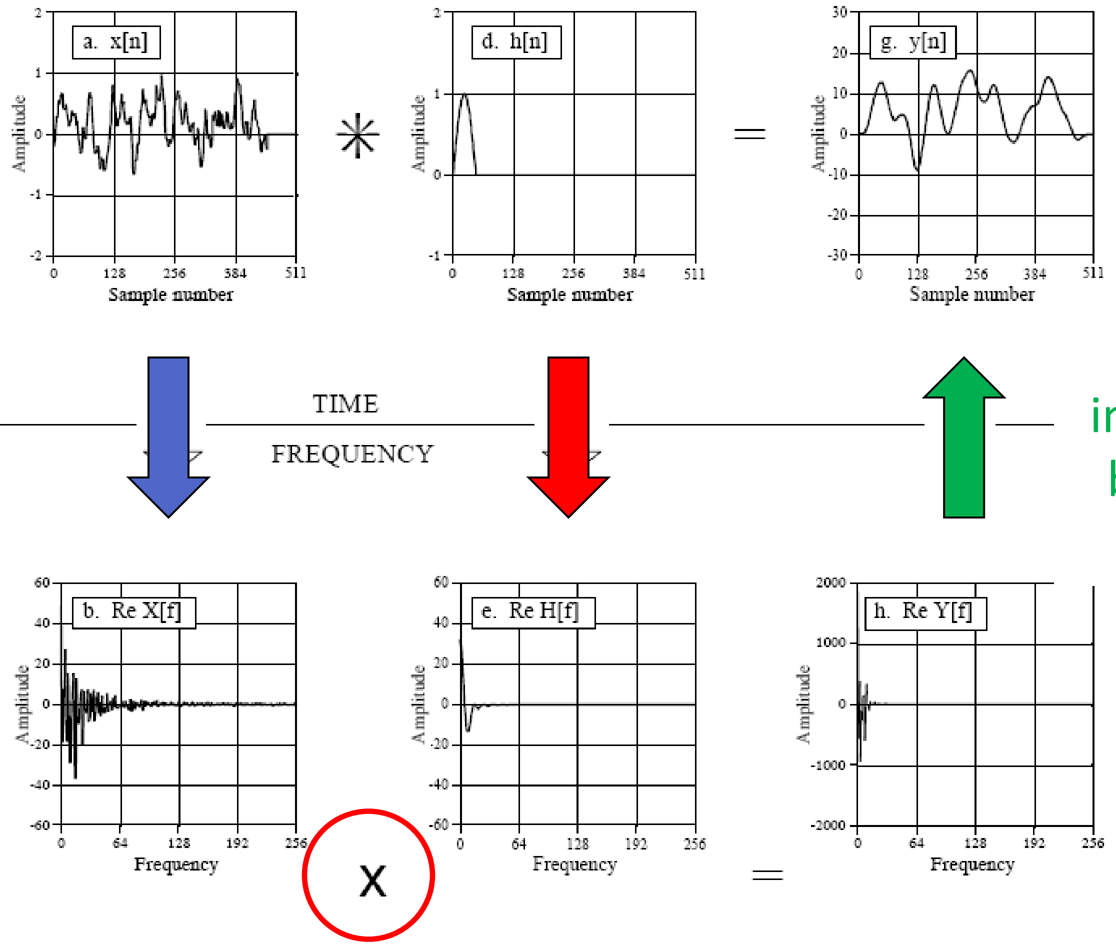
$$y[n] = x[n] * h[n]$$

y has length $M + N - 1$

DFT Convolution

Take the DFT of the input sequence

And the DFT of the impulse response

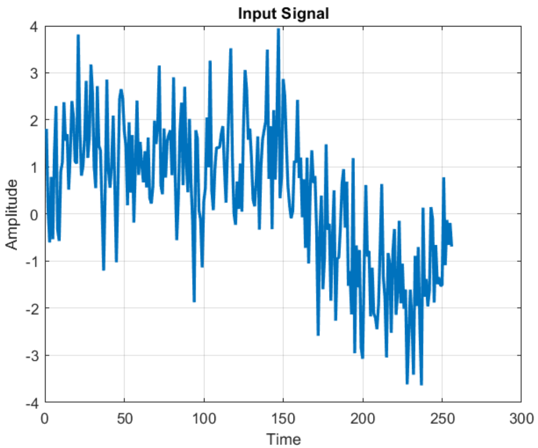


Finally take the inverse DFT to get back to the time domain

Multiply the DFT together point by point

Time Domain Convolution and Frequency Domain Multiplication

Time Domain



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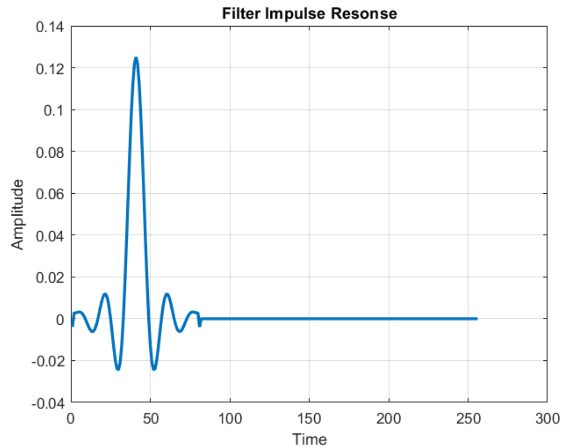

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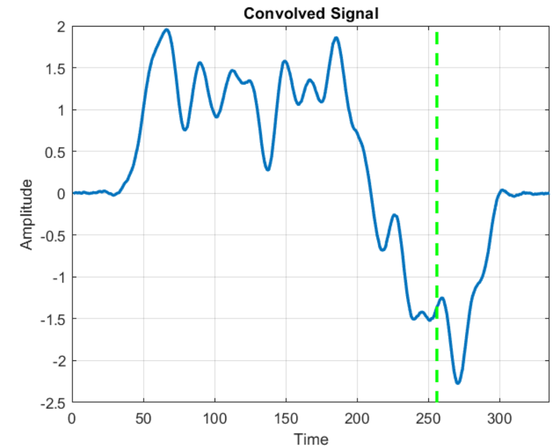
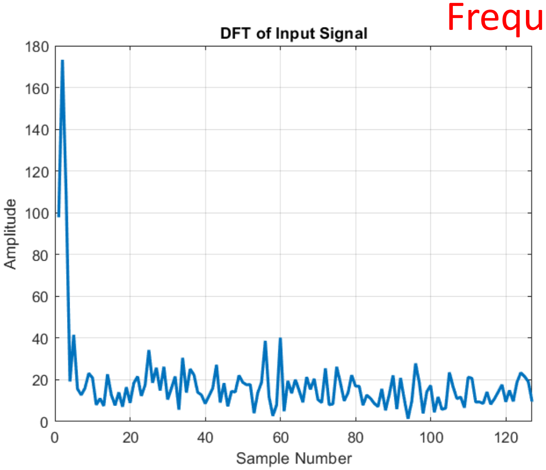


M = 256

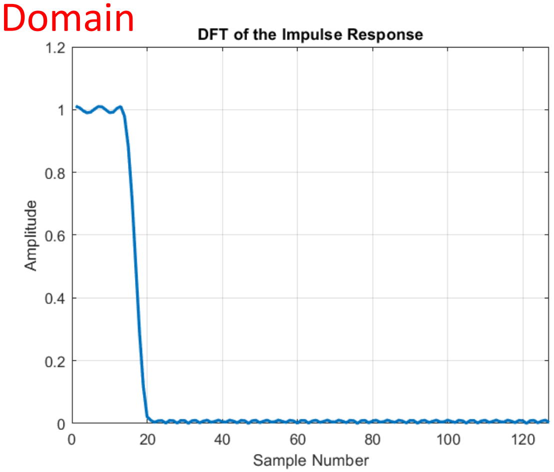
N = 81

M+N-1 = 336

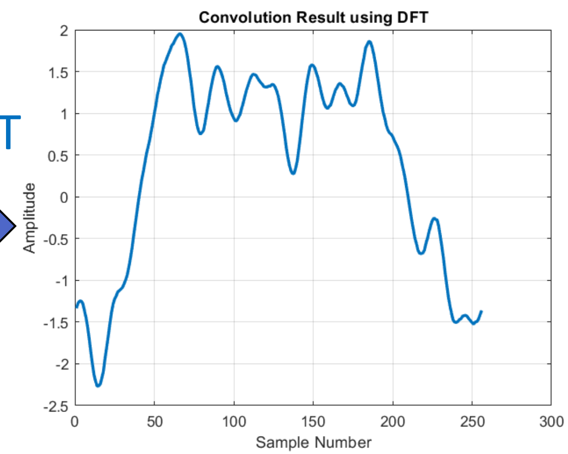
Frequency Domain



X



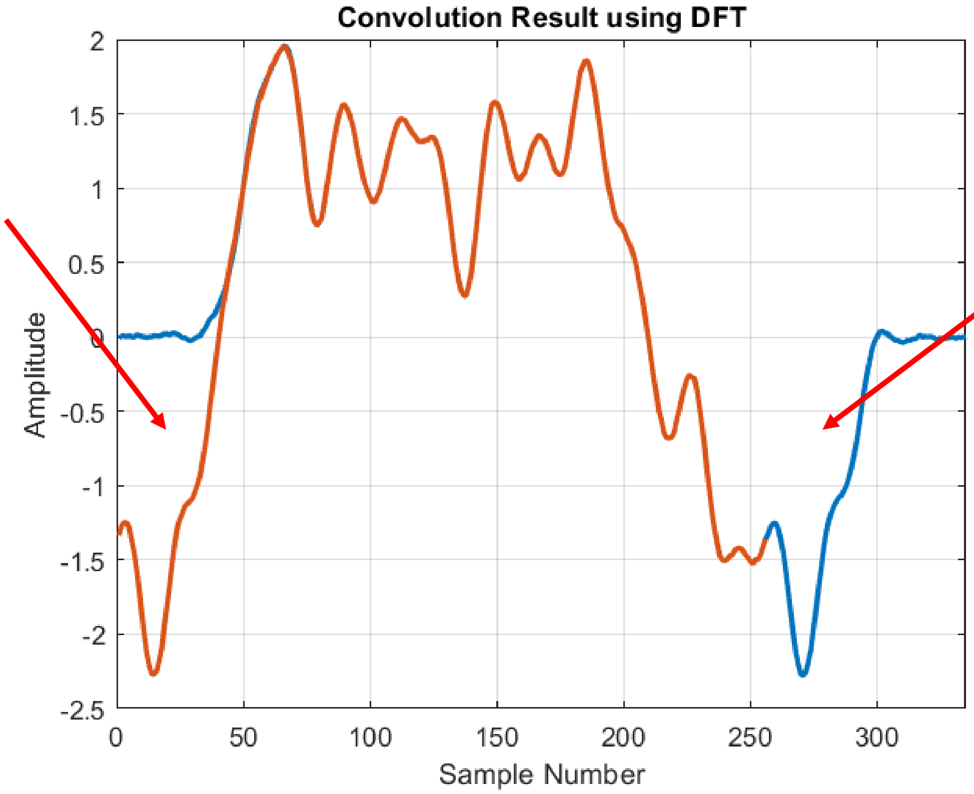
IDFT



Time Domain Convolution and Frequency Domain Multiplication

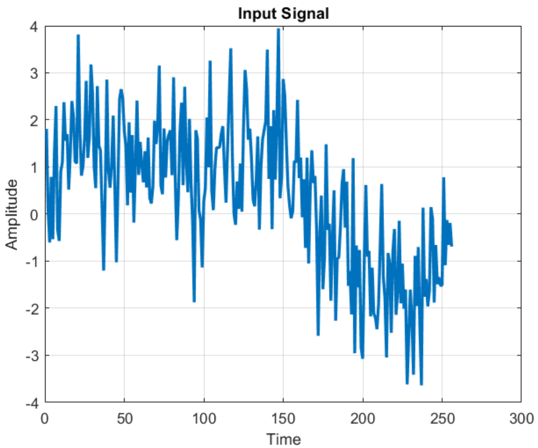
- The two signals should match, but do not. Why?

Distorted signal at the beginning



Missing signal at the end

Circular Convolution



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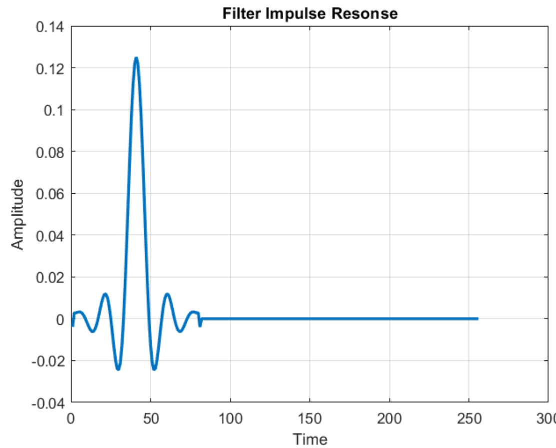

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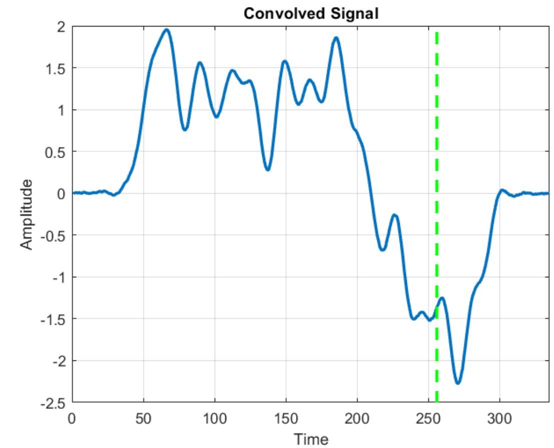
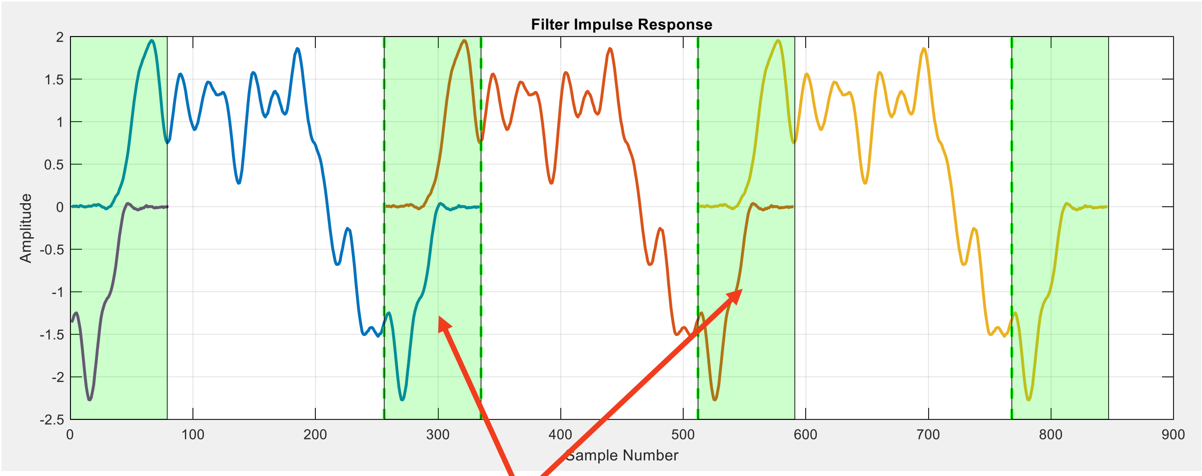


The length of the convolution output is 336.

The length of the FFT multiplication is only 256

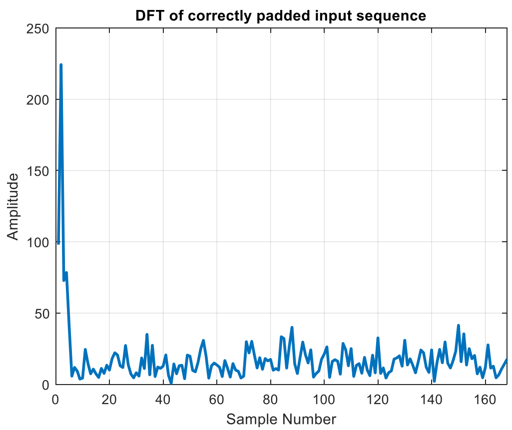


There is overlap of the two sequences causing distortion

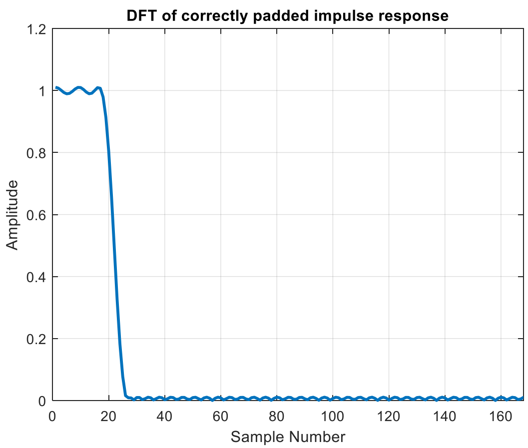
Frequency Domain Convolution With Correct Padding

- After correctly padding the input signals to M+N-1 samples, the IDFT output matches the output of the time domain convolution

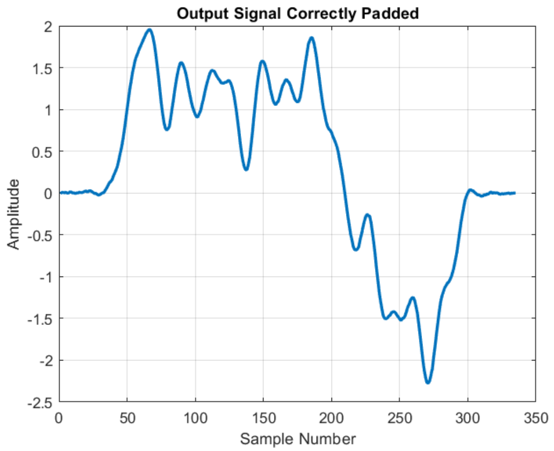
Frequency Domain Output



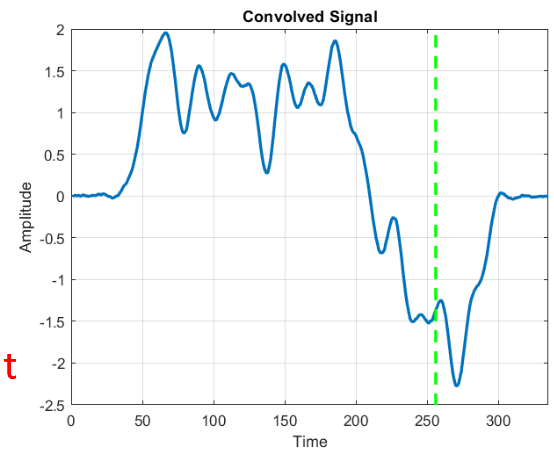
X



IDFT



Padding the input sequence and the impulse response to length 336 results in a correct convolution output



Time Domain Output

Multiplication in the Time Domain

- Convolution in the time domain is equivalent to multiplication in the frequency domain

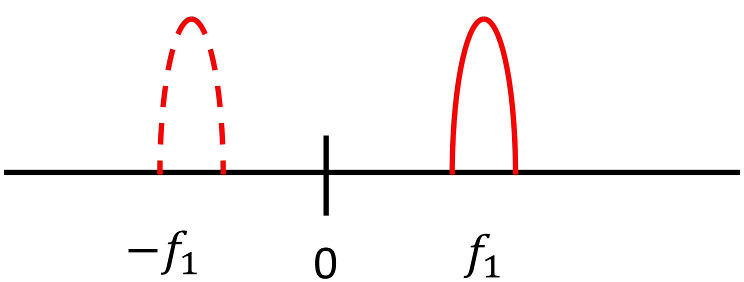
$$x[n] * y[n] \;\; \leftrightarrow X[k] \times Y[k]$$

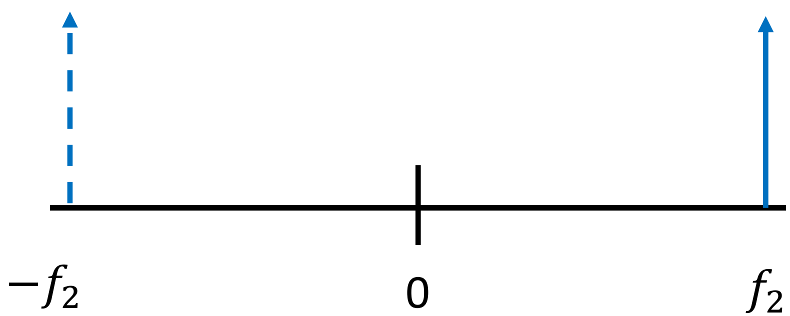
- <u>Multiplication</u> in the <u>time</u> domain is equivalent to convolution in the frequency domain

$$x[n] \times y[n] \;\; \leftrightarrow X[k] * Y[k]$$

What is Convolution in the Frequency Domain?

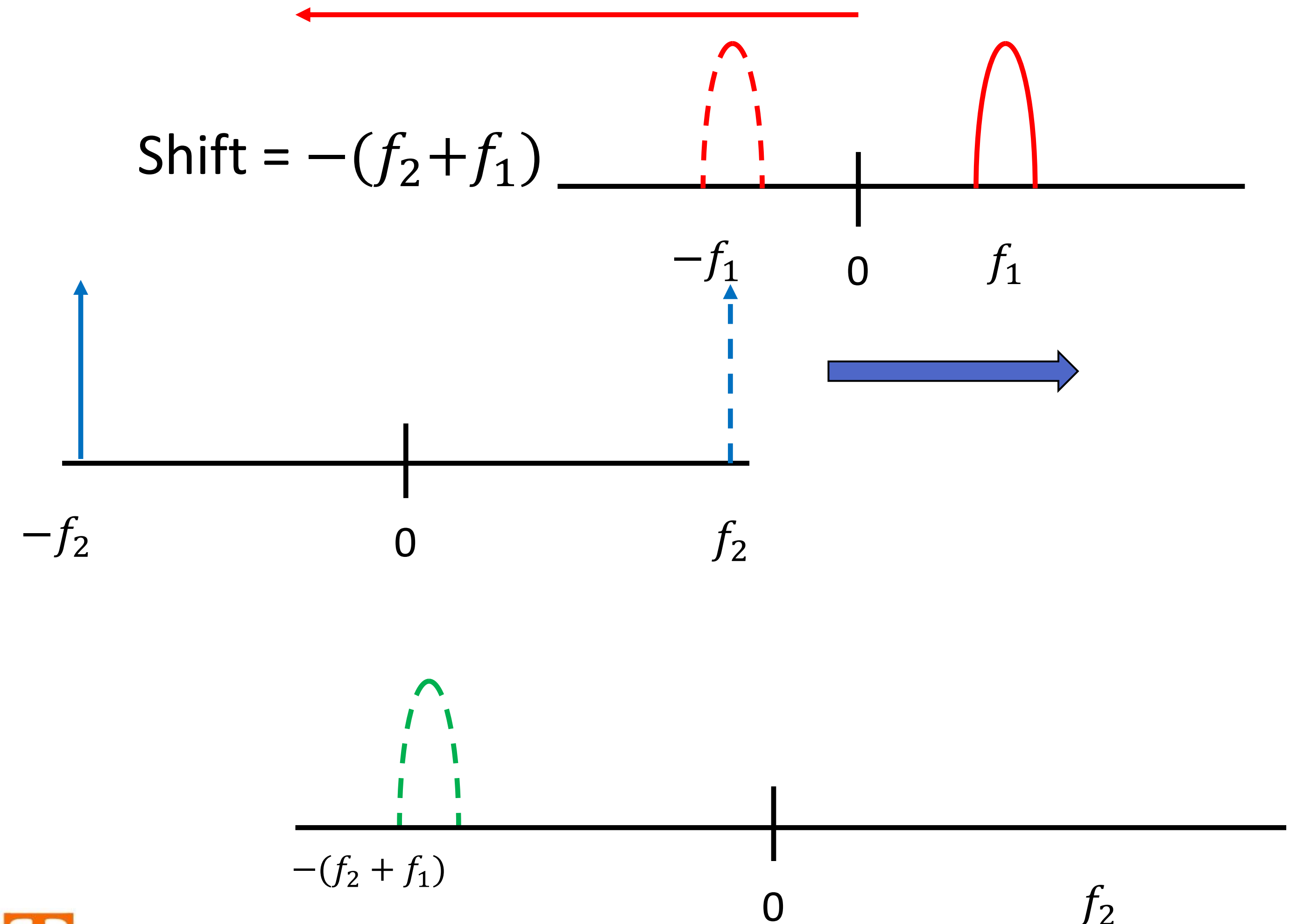
- Suppose I multiply two real signals centered at f_1 and f_2
- Recall that each has a negative frequency component as well



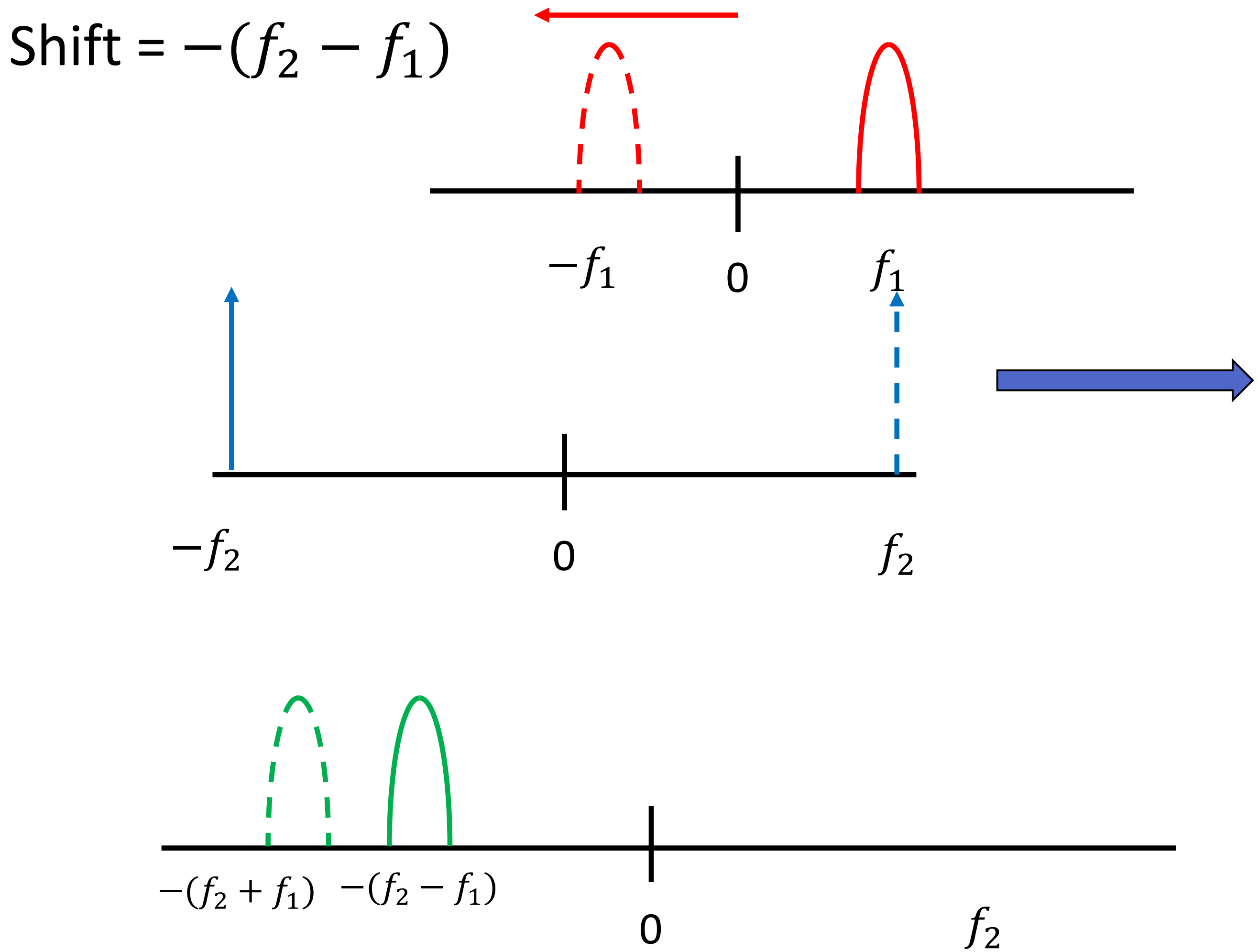


- Multiplying these in the time domain is convolving them in the frequency domain

What is Convolution in the Frequency Domain

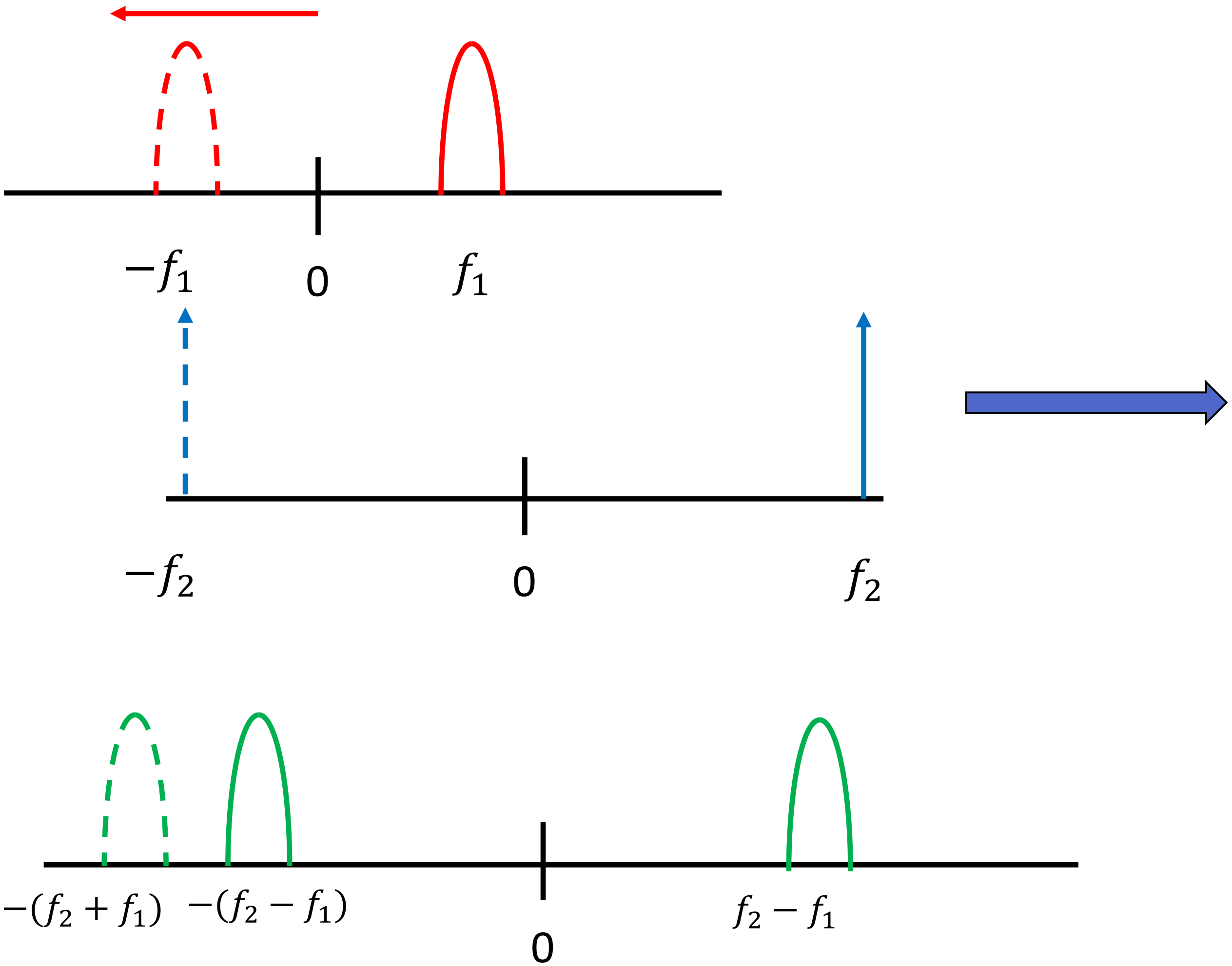


What is Convolution in the Frequency Domain

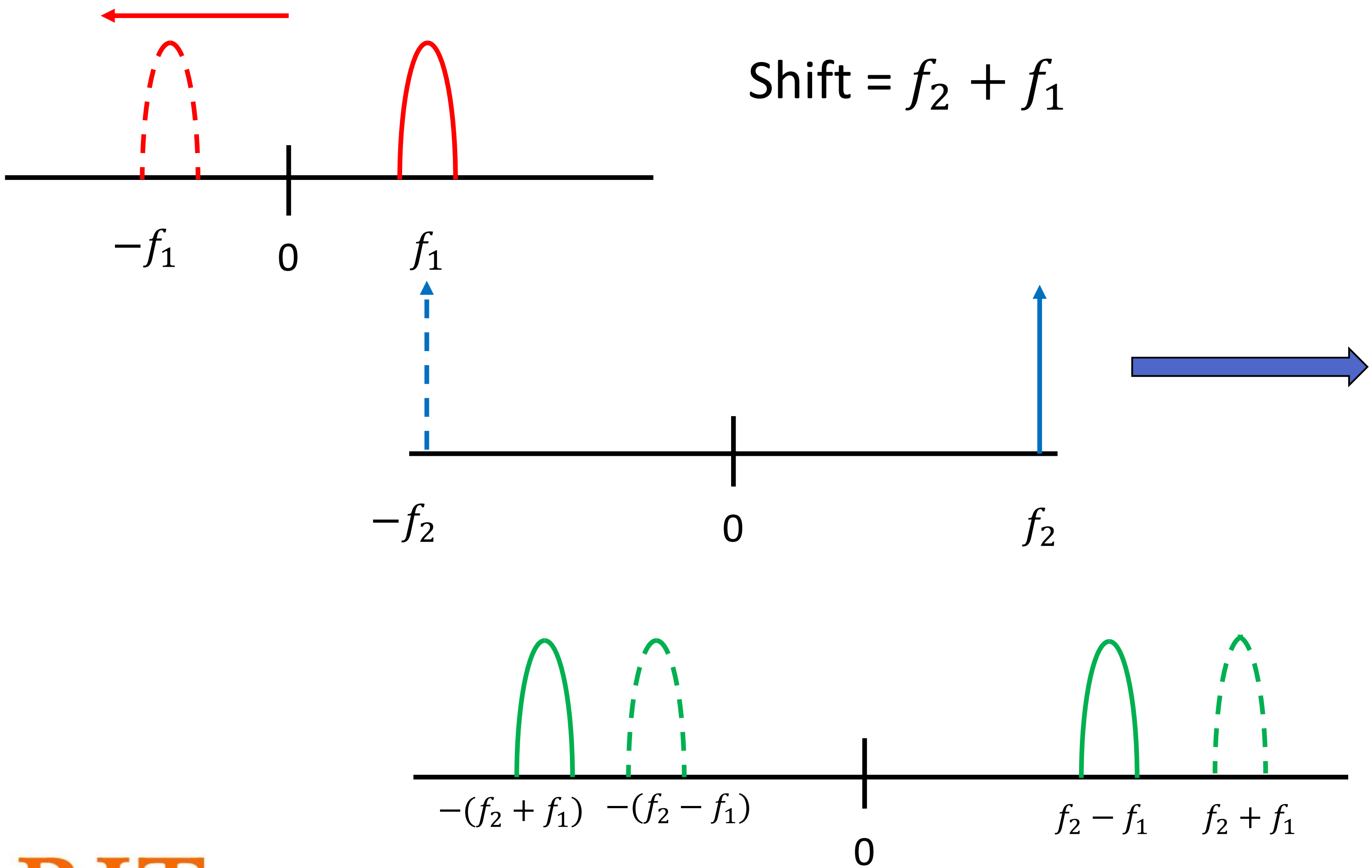


What is Convolution in the Frequency Domain

Shift = $f_2 - f_1$



What is Convolution in the Frequency Domain

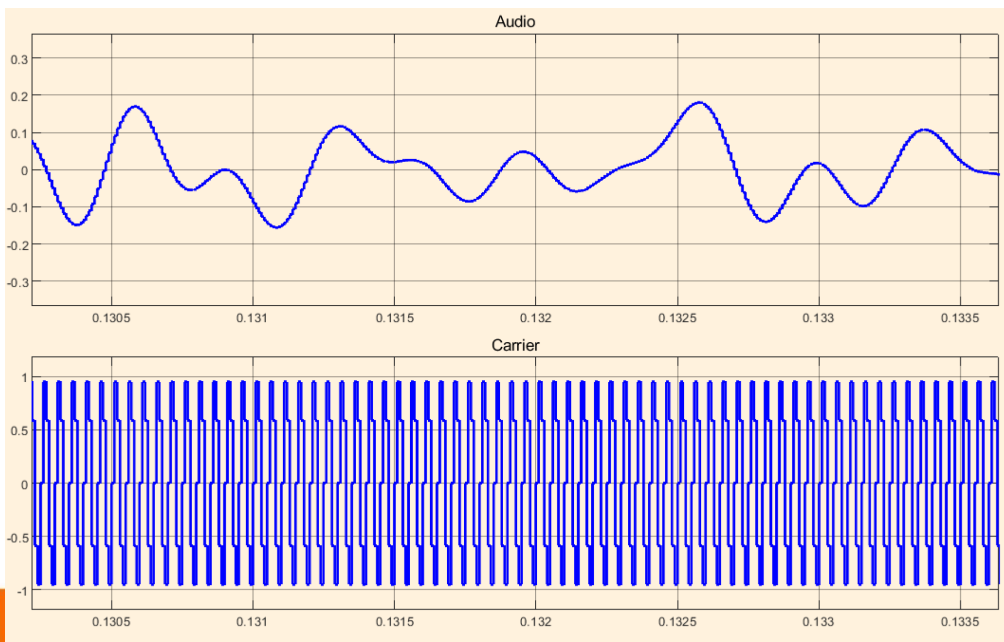


Convolution in Frequency Domain

- Where does this property come into play?
- A good example of this is in AM or DSB modulation.
- The information signal (generally audio) is multiplied in the time domain by a higher frequency carrier.
- This is one way to create AM modulation

AM Modulation

- Suppose I have a low frequency audio signal from 300 Hz to 3000 Hz
- I multiply that by a sine wave (“carrier”) at 20 kHz



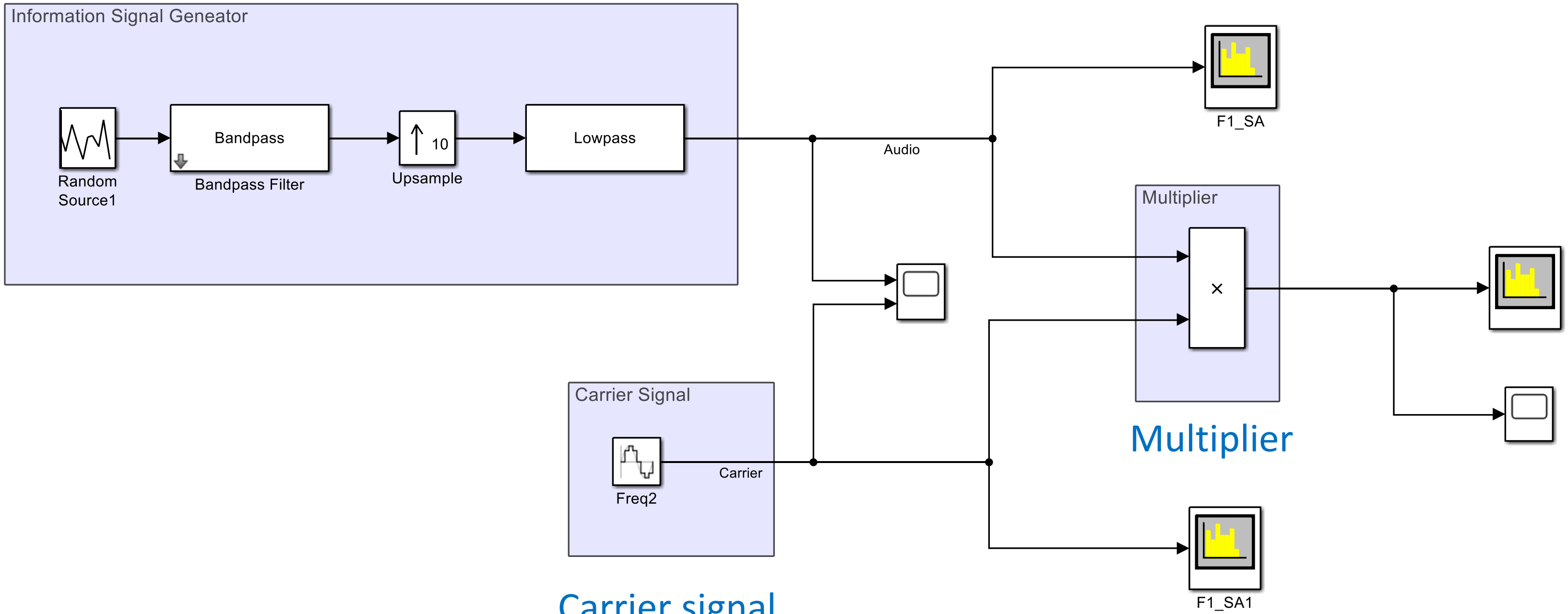
Low Frequency information (audio) signal

Higher frequency sine wave or carrier

AM/DSB Modulation Simulation

- Block Diagram of the AM/DSB modulation simulation

Information Generator

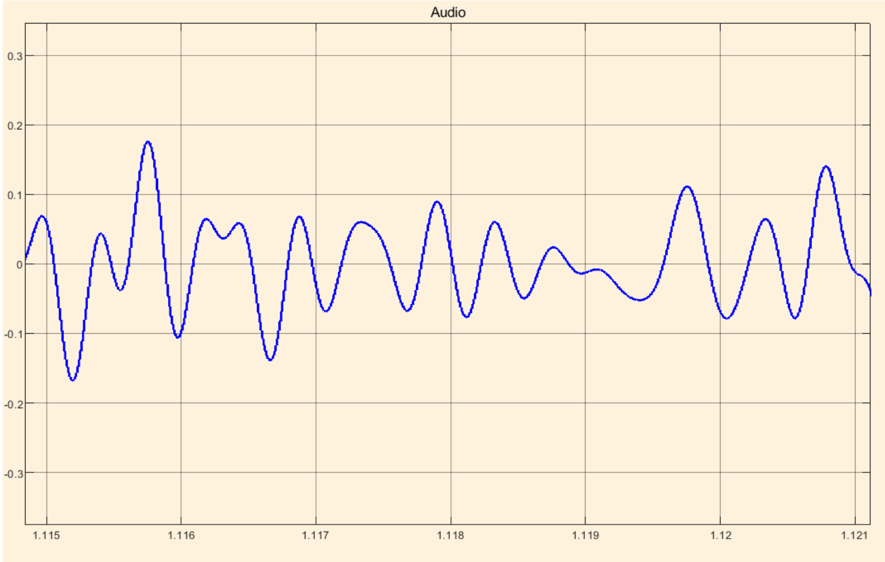


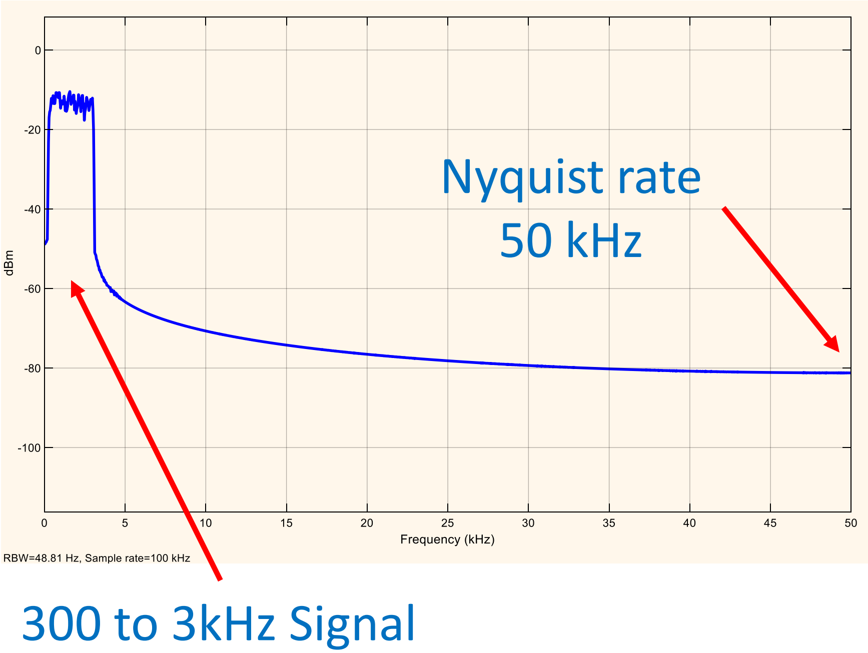
Multiplier

Carrier signal
(20 kHz Sinewave)

The Information Signal

- The information signal in this example simulates and audio signal
- It is a bandlimited signal with frequency content from 300 Hz to 3000Hz

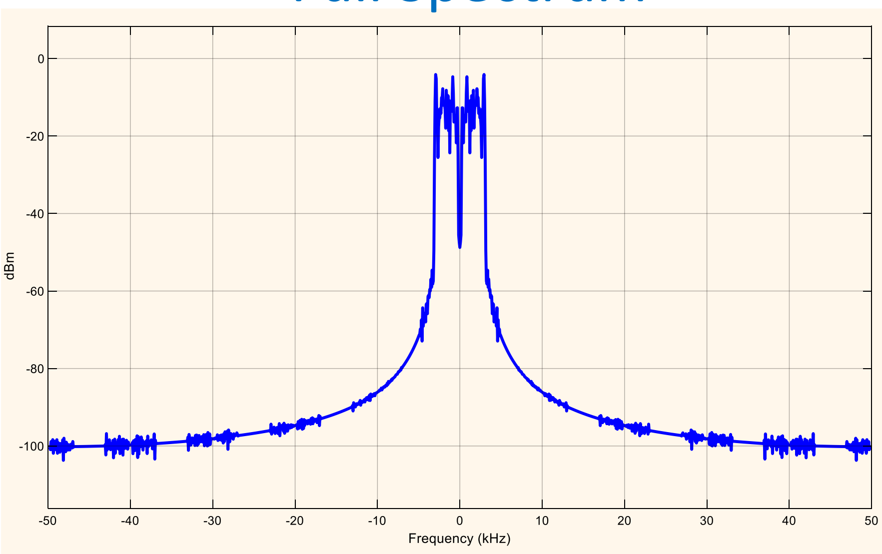




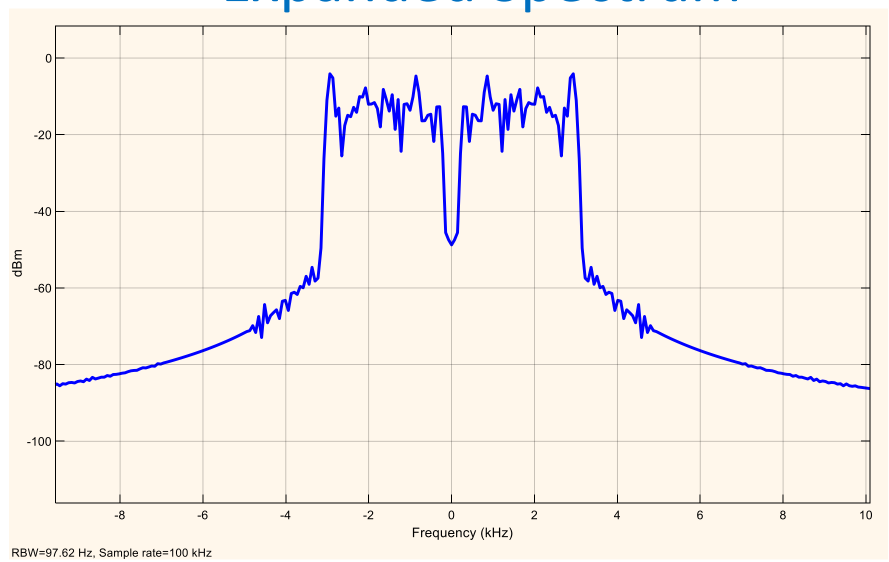
Information Signal

- The audio information has both a positive and a negative frequency component.

Full Spectrum



Expanded Spectrum

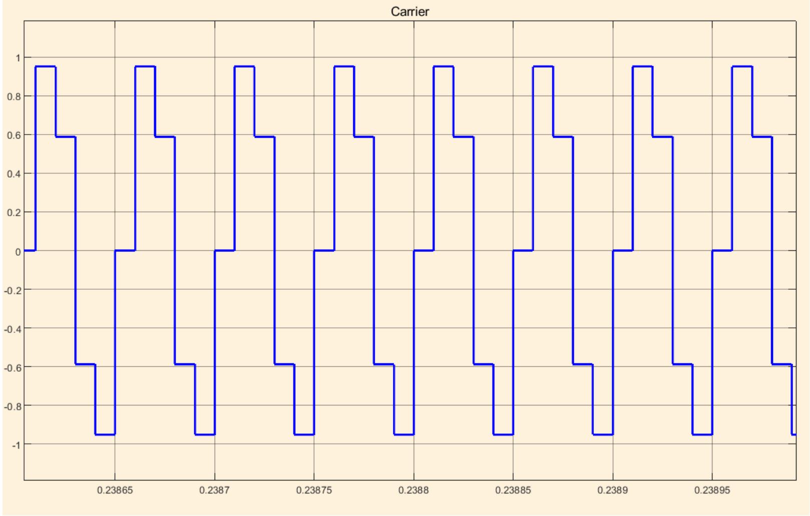


Spectrum is shown centered around 0 Hz $\pm 50\ kHz$

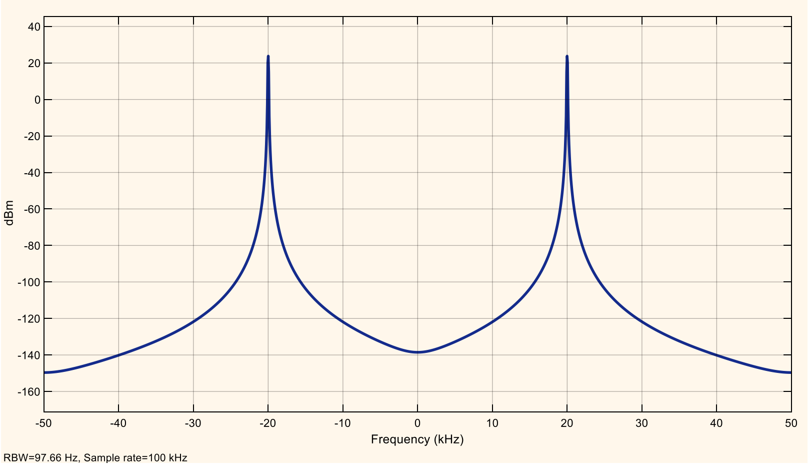
The Carrier Signal

- The carrier signal is a single frequency sine wave
- We are simulating using 20 kHz, but typically the frequency would be much higher from 1 MHz to 100's of MHz

Time Domain



Frequency Domain

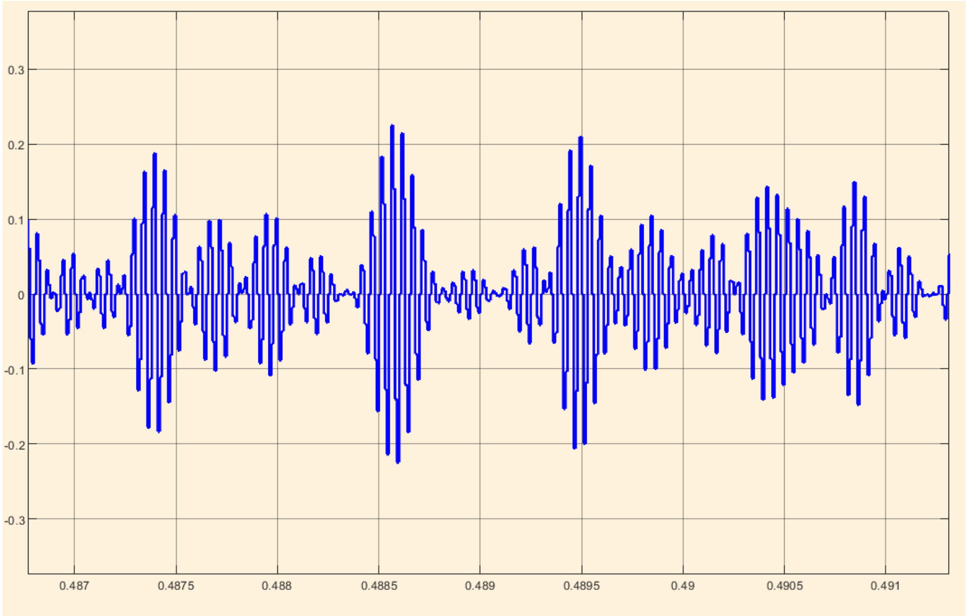


Positive and Negative frequency spectra

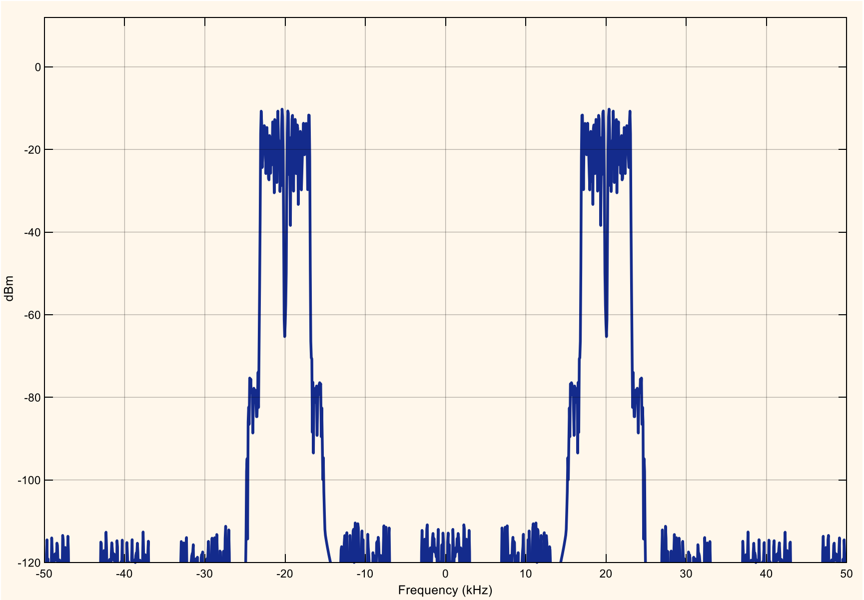
Modulation Result

- The AM Spectra (double sided)
- The information signal is copied to the location of the carrier signals

Time Domain

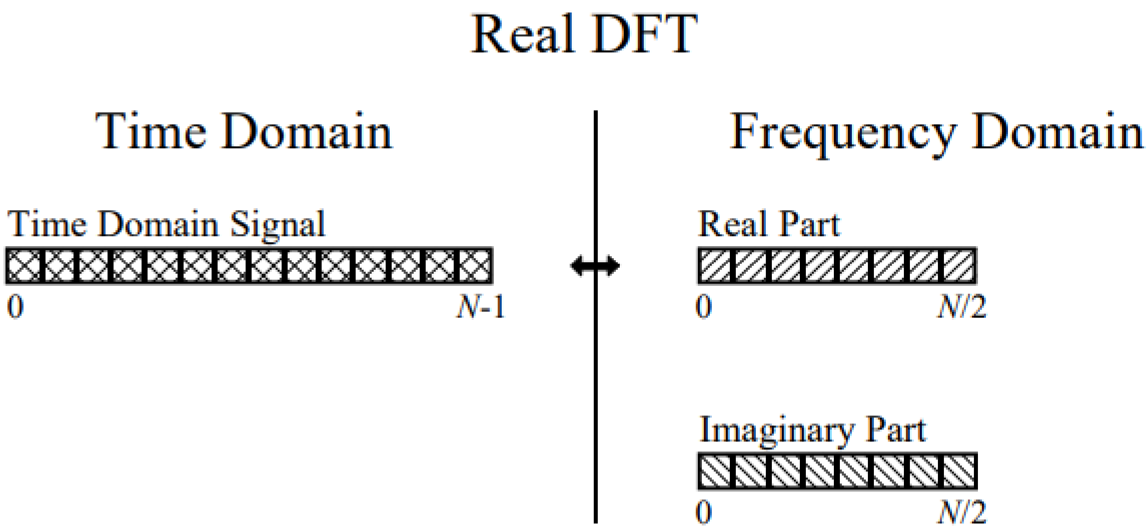


Frequency Domain



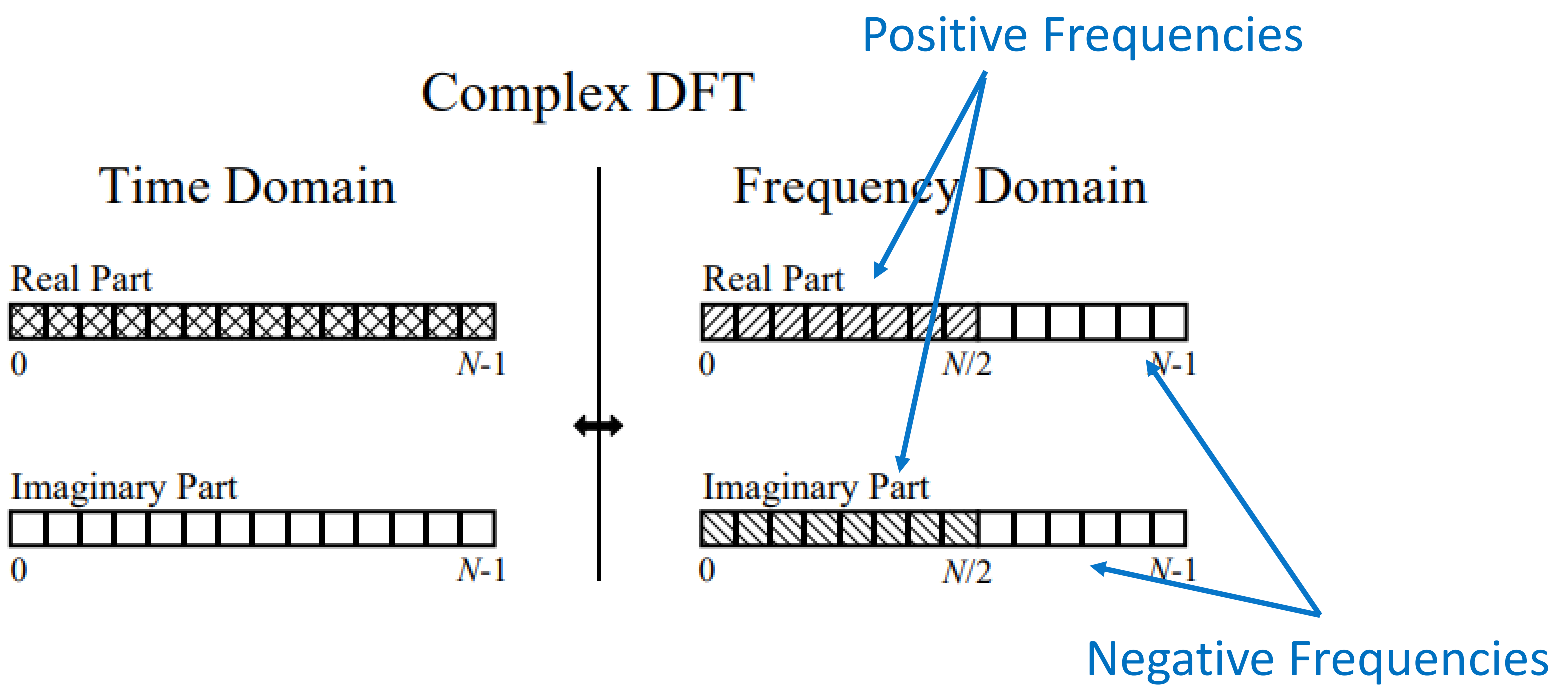
Complex and Real DFT

- For the Real DFT we took a real time domain sequence of length N and computed a real and imaginary sequence in the frequency domain of length $\frac{N}{2} + 1$.



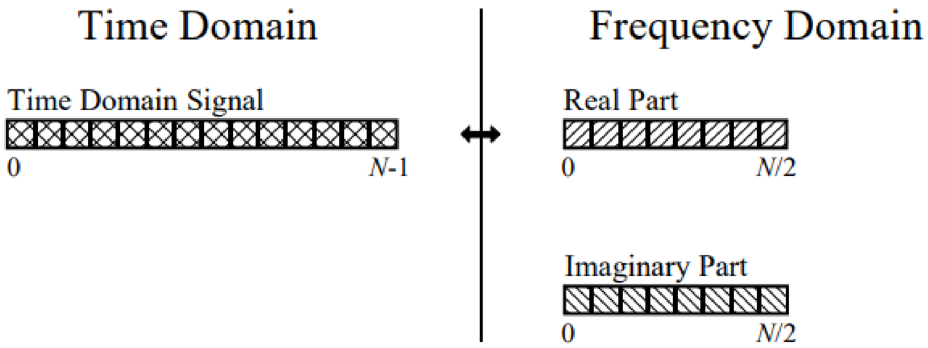
Complex and Real DFT

- The complex DFT takes a complex time domain signal of length N and computes a complex frequency domain signal of length N



Complex and Real DFT

Real DFT

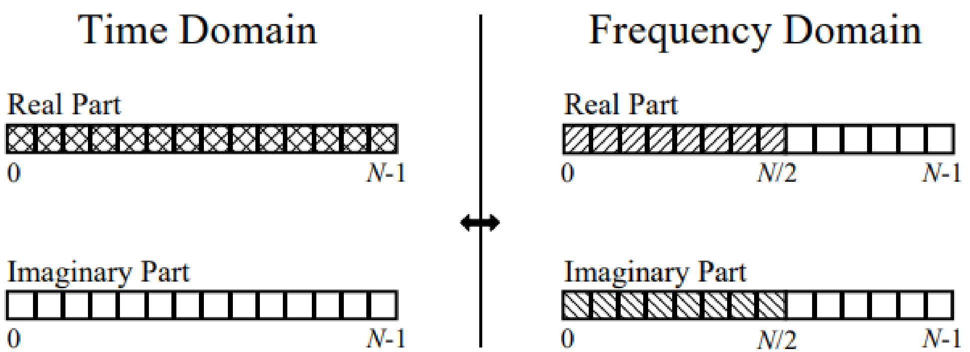


$$Re(X[k]) = \sum_{i=0}^{N-1} x[i] \cos\left(\frac{2\pi ki}{N}\right)$$

$$Im(X[k]) = \sum_{i=0}^{N-1} x[i] \sin\left(\frac{2\pi ki}{N}\right)$$

For $k = 0 \; to \; \frac{N}{2}$

Complex DFT

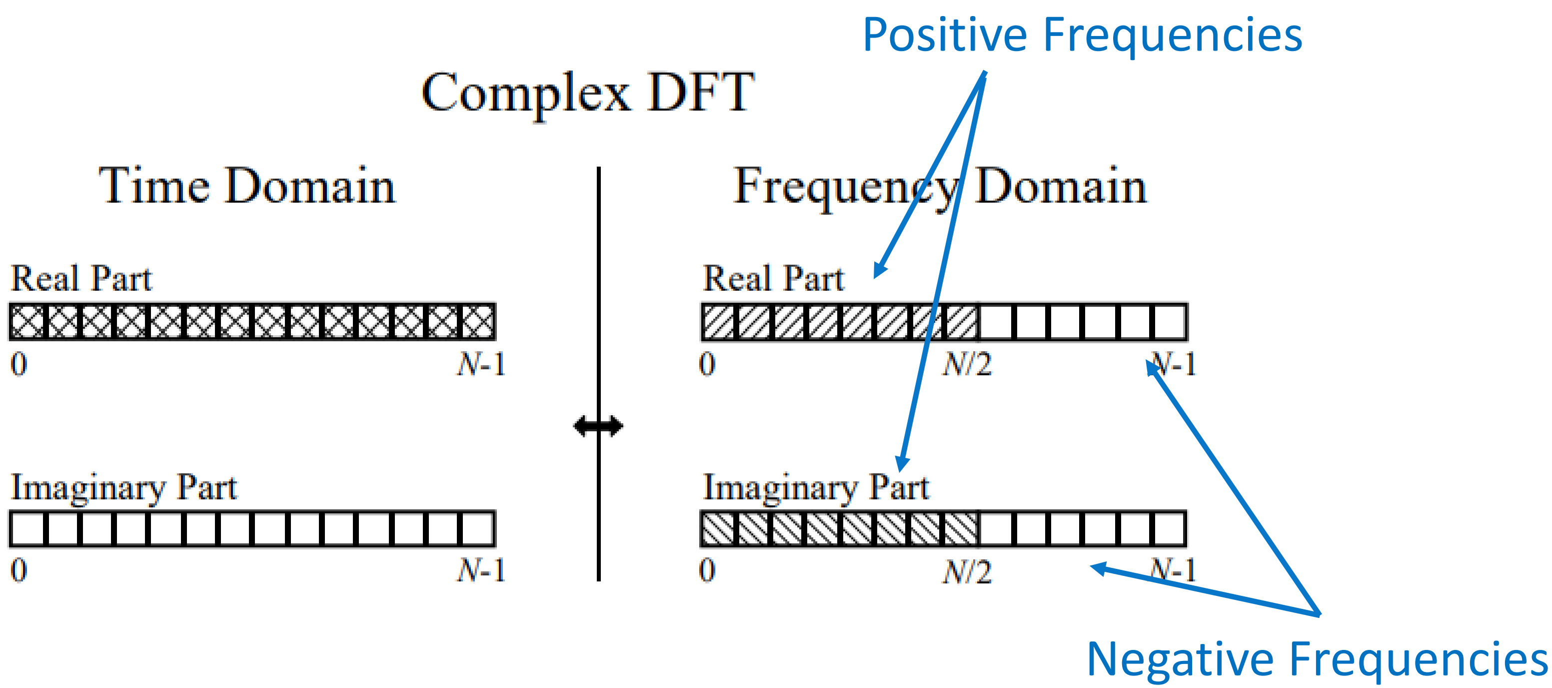


$$X[k] = \sum_{i=0}^{N-1} x[i] e^{-\frac{j2\pi ki}{N}}$$

For $k = 0 \; to \; N - 1$

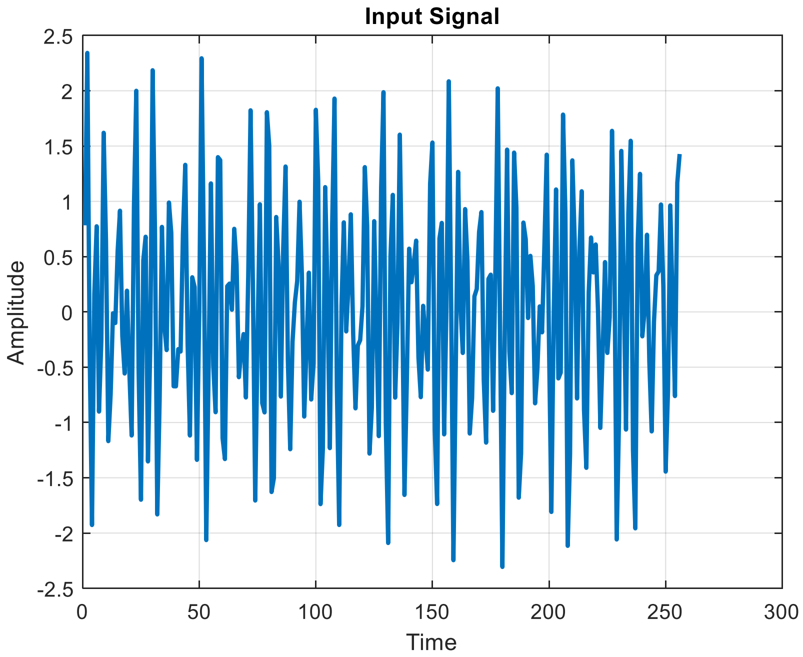
Compute the Real DFT using the FFT

- Set the imaginary part of the sequence to all zeros



FFT Symmetry

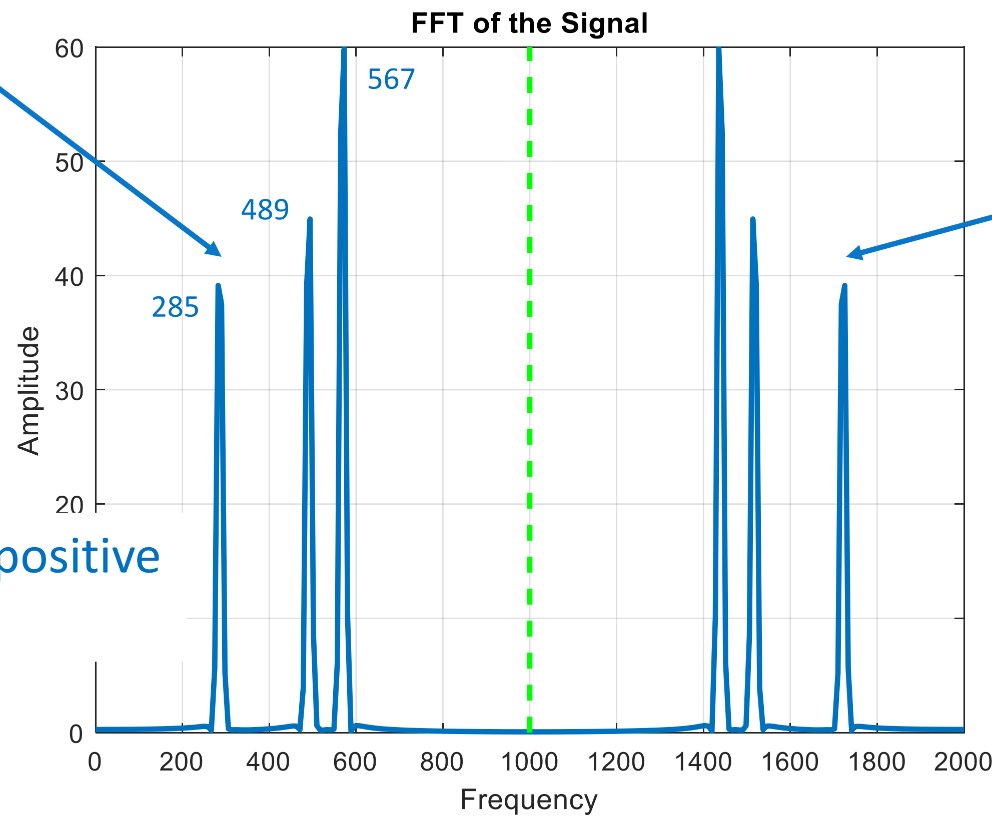
- Compute the FFT of a signal with three sinusoids (285, 489, 567 Hz)



FFT Symmetry

Positive Frequencies

Negative Frequencies Repeated at the sample Rate

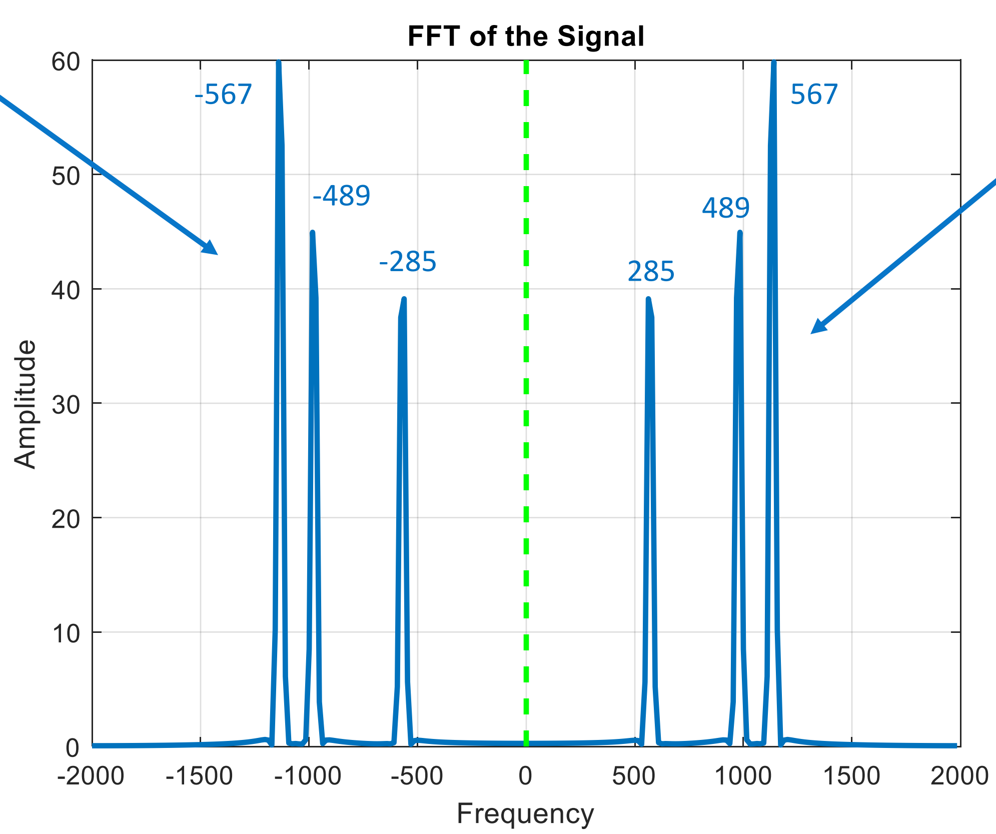


DFT shows just positive frequencies

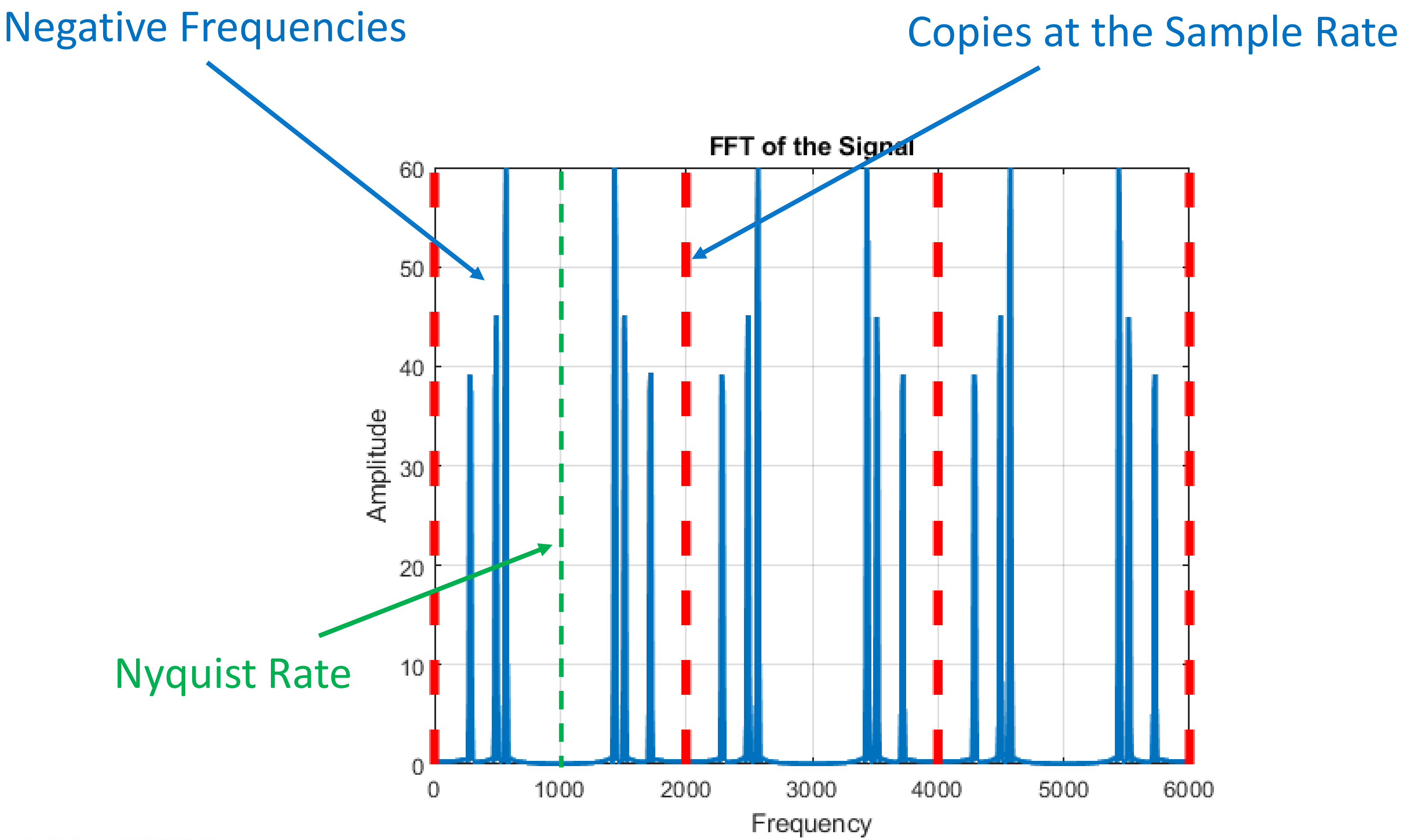
FFT Symmetry
Plot 0 Frequency at Center

Negative Frequencies

Positive Frequencies



FFT Symmetry
Plot Over Extended Frequency



Using the FFT to Compute the Real DFT

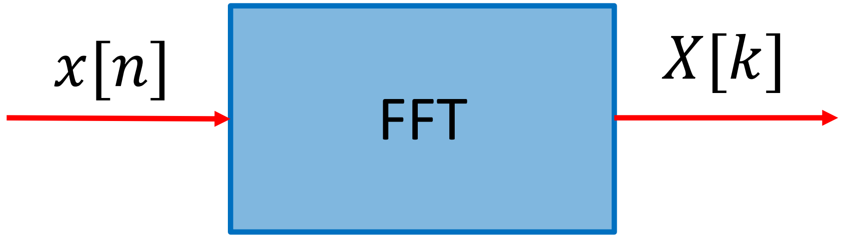
- The FFT can be used to compute the Real DFT
- For a real time domain sequence the imaginary part is all zero
- The values of the real DFT are the first $\frac{N}{2} + 1$ samples of the FFT
- The FFT includes both positive and negative frequency components

Other Fourier Transform Properties

Linearity of the Fourier Transform

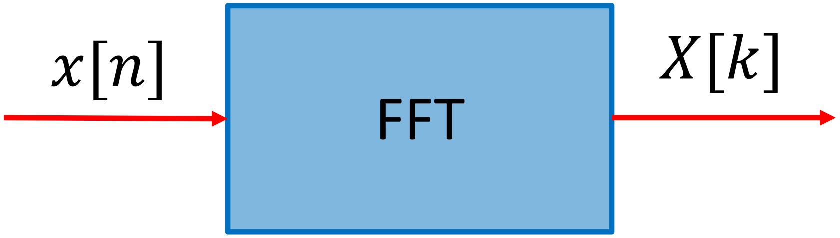
- We can think of the Fourier Transform as a system

- An input sequence goes into the system and we get an output sequence.



Linearity of the Fourier Transform

- The DFT is linear
  - It has homogeneity
  - It has additivity
- The DFT does not have time shift invariance

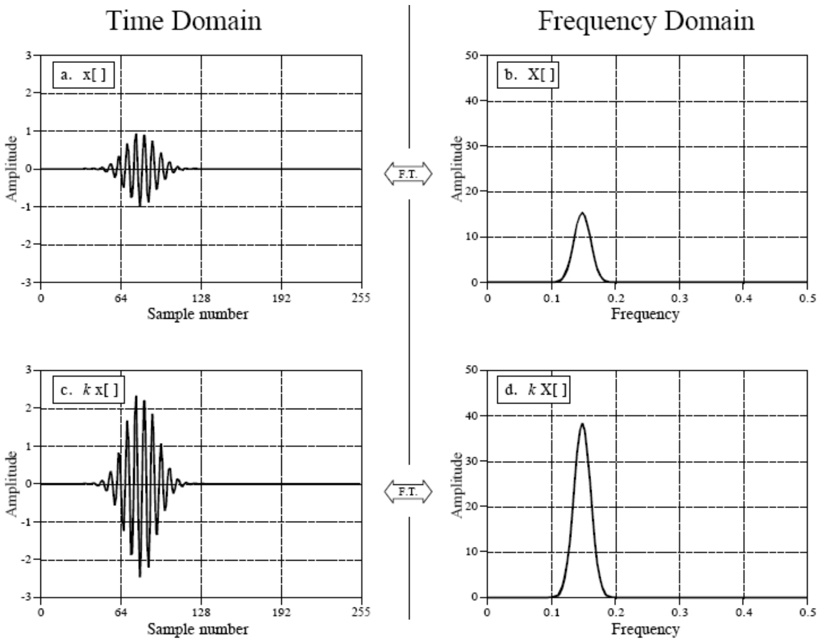


Homogeneity of the Fourier Transform

- Recall what homogeneity means
- If we increase the amplitude of the input signal, the output of the system increases proportionally
- Mathematically

$$x[n] \Longleftrightarrow X[k]$$

$$kx[n] \Longleftrightarrow kX[k]$$



Additivity of the Fourier Transform

- Recall what additivity is:

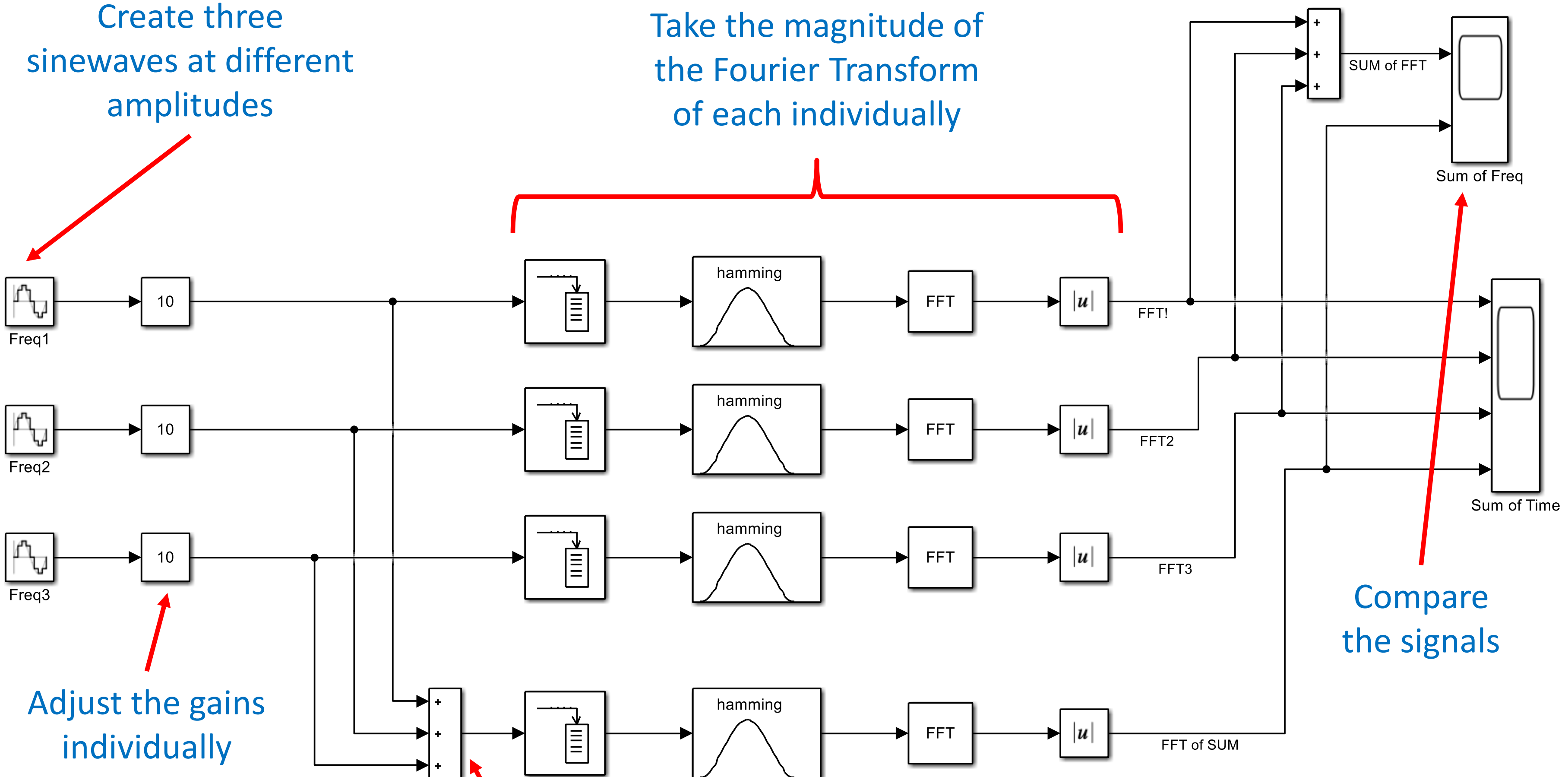
- The output from the system of the sum of two inputs will be equal to the sum of the individual outputs from two inputs
- Mathematically

$$f(x_1[n] + x_2[n]) = f(x_1[n]) + f(x_2[n])$$

Fourier Transform Linearity

Create three sinewaves at different amplitudes

Take the magnitude of the Fourier Transform of each individually



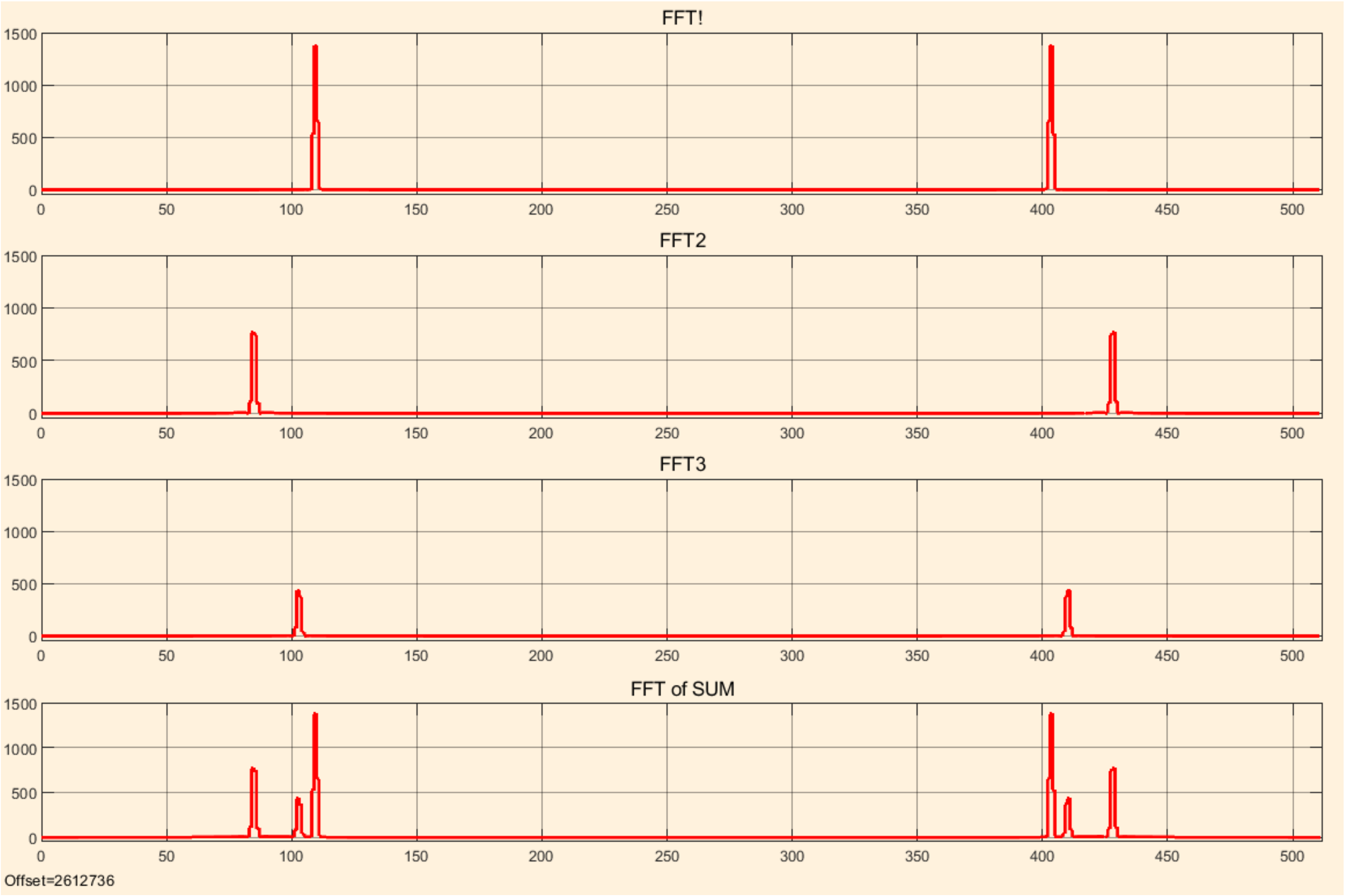
Compare the signals

Adjust the gains individually

Sum all the signals and take the Fourier Transform

$f(x_1 + x_2 + x_3)$

Adding Each Individual DFT Homogeneity



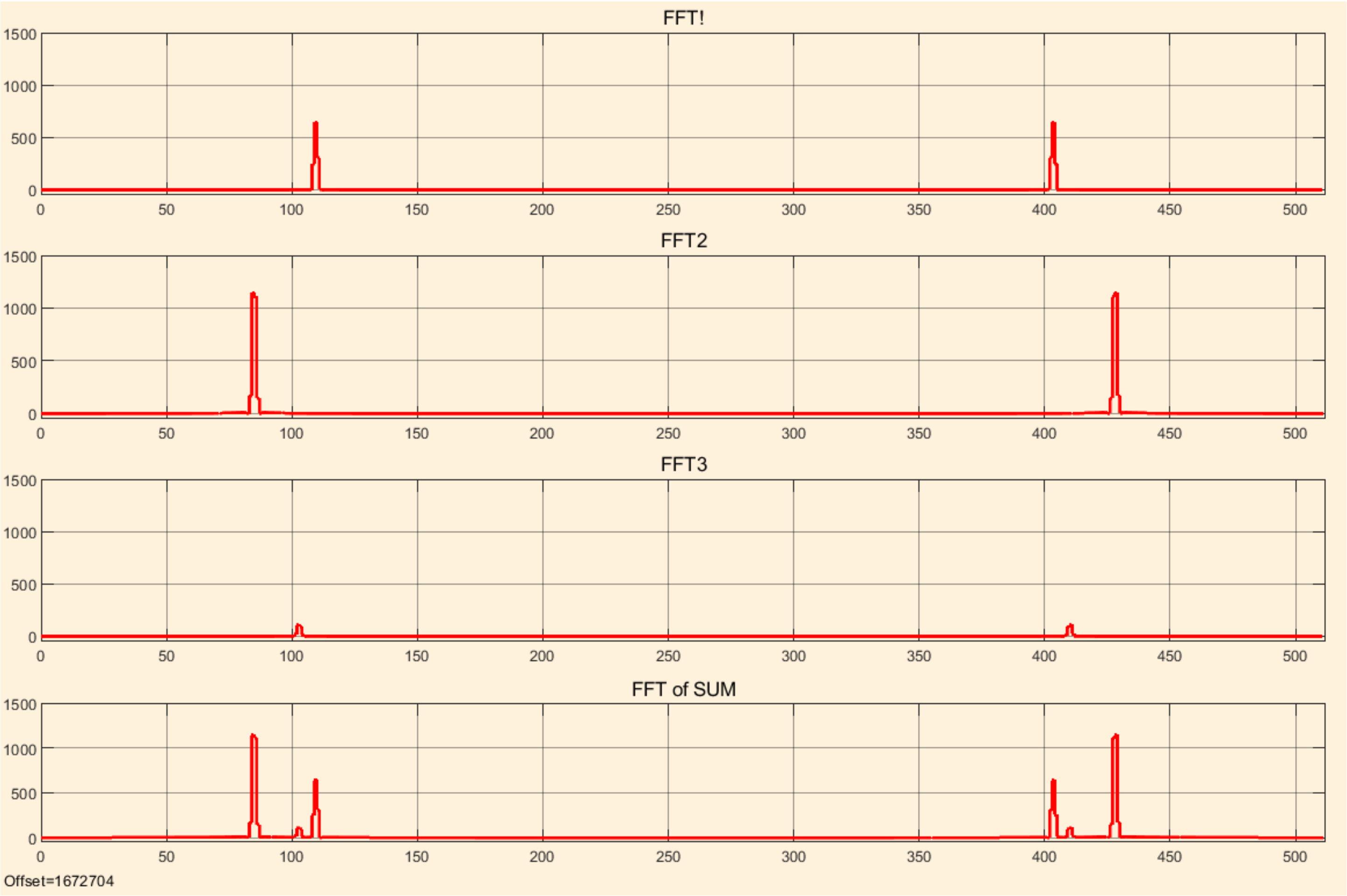
FFT1 (magnitude)

FFT2 (magnitude)

FFT3 (magnitude)

Sum of FFT’s (magnitude)

Changing Input Levels Change the Output Levels



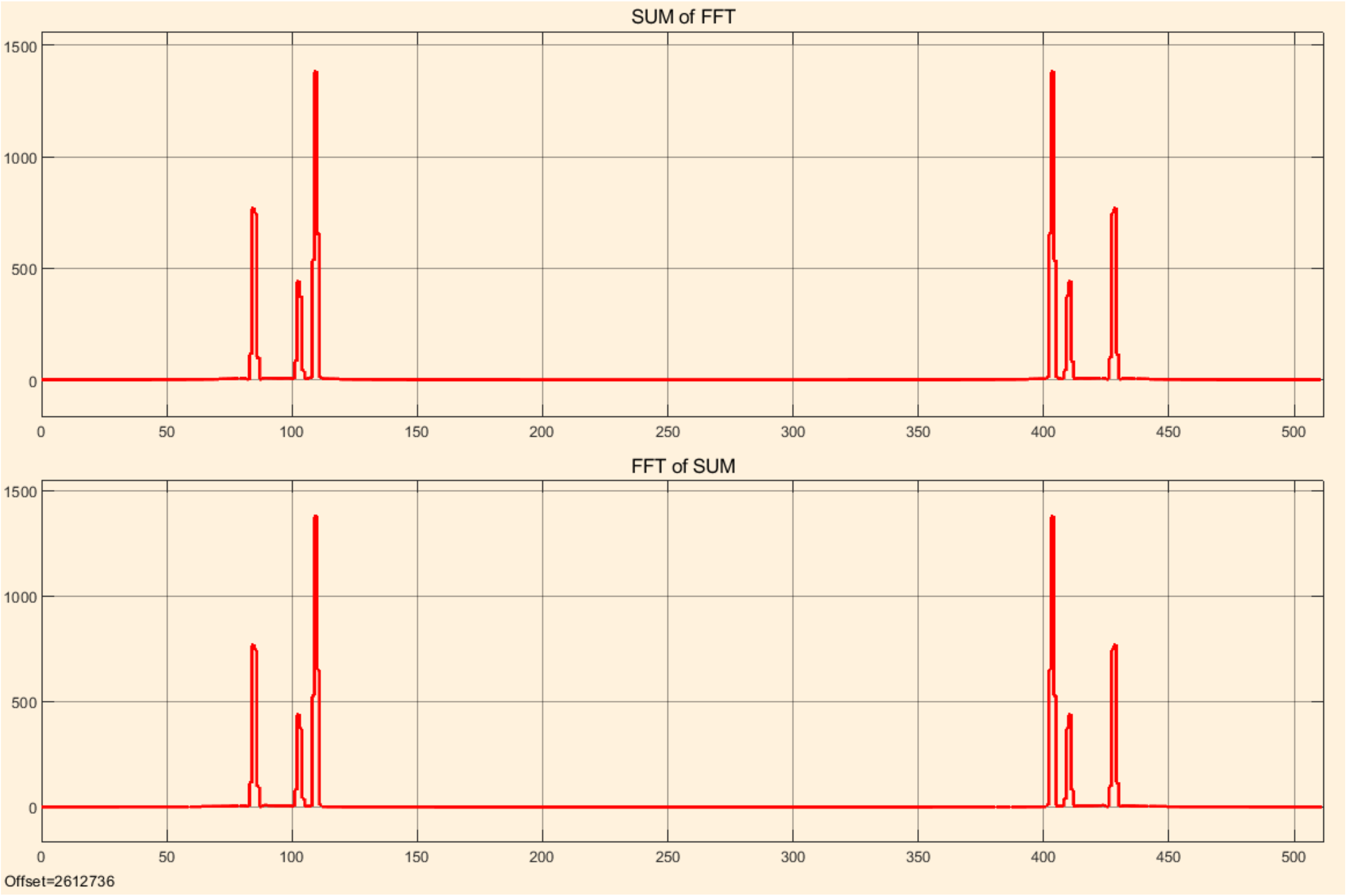
FFT1 (magnitude)

FFT2 (magnitude)

FFT3 (magnitude)

Sum of FFT's (magnitude)

Compare Sum of FFT and FFT of SUM



SUM of FFT (Magnitude)

FFT of SUM (Magnitude)

Additivity of the Fourier Transform

- Note that additivity of the Fourier transformed signals is valid for adding the signals using the real and imaginary terms
- Additivity of the signals does not work when just adding the magnitude terms.
- You can’t directly add values in polar format. You must use the real and imaginary form.

Characteristics of the Phase of the Fourier Transform

- The Fourier Transform is <u>not</u> time shift invariant.

- A time shift in the time domain does <u>not</u> equal a time shift in the frequency domain.

$$x[n] \Longleftrightarrow |X[k]| \angle \Theta(k)$$

Time Domain

Frequency Domain (Magnitude and Phase)

Characteristics of the Phase of the Fourier Transform

- A *time* shift of the input adds a linear (with frequency) *phase* shift to the output

- The *magnitude* stays the same

Linear phase shift with frequency

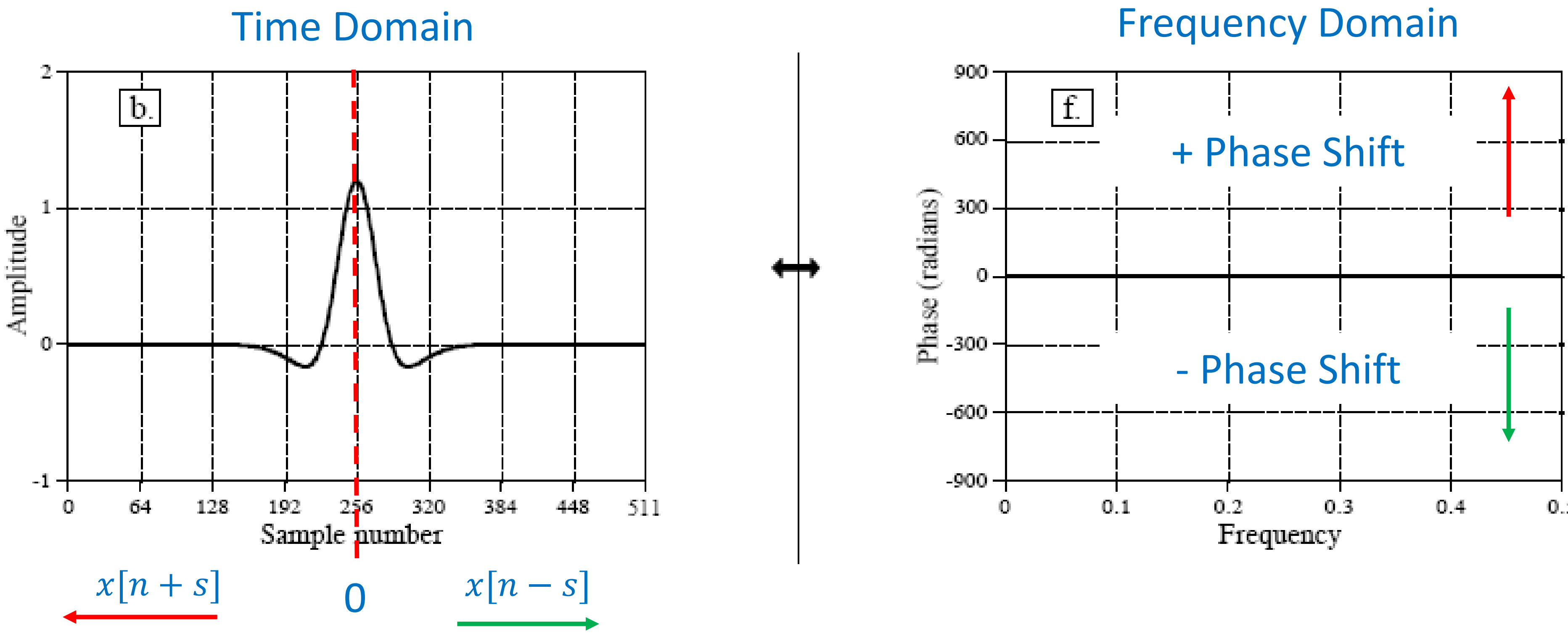
$$x[n+s] \Longleftrightarrow |X[k]| \angle \Theta(k) + 2\pi s f$$

Time Domain

Frequency Domain (Magnitude and Phase)

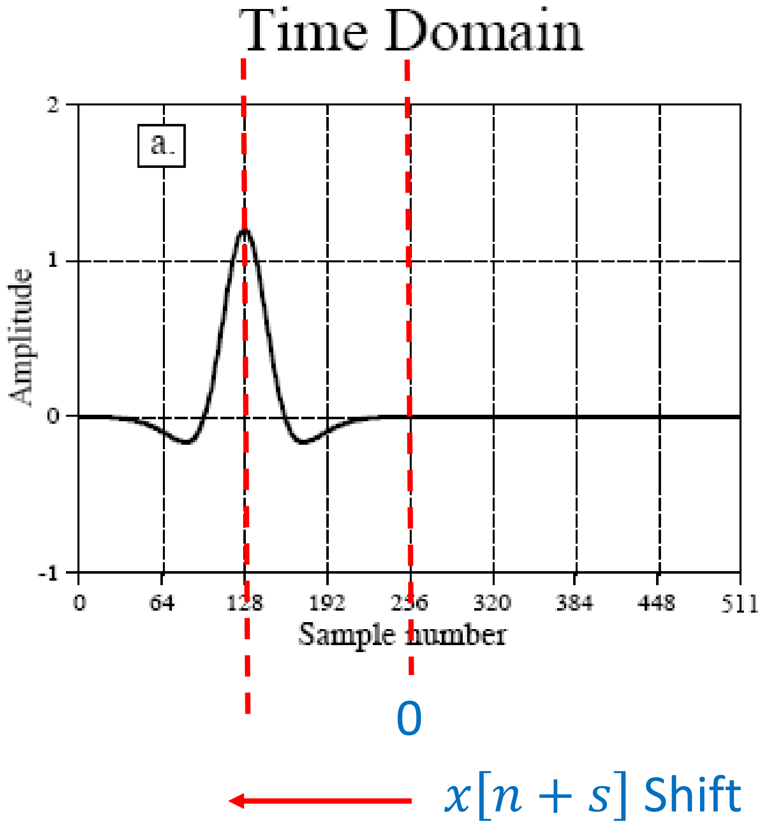
Time Shift Illustration

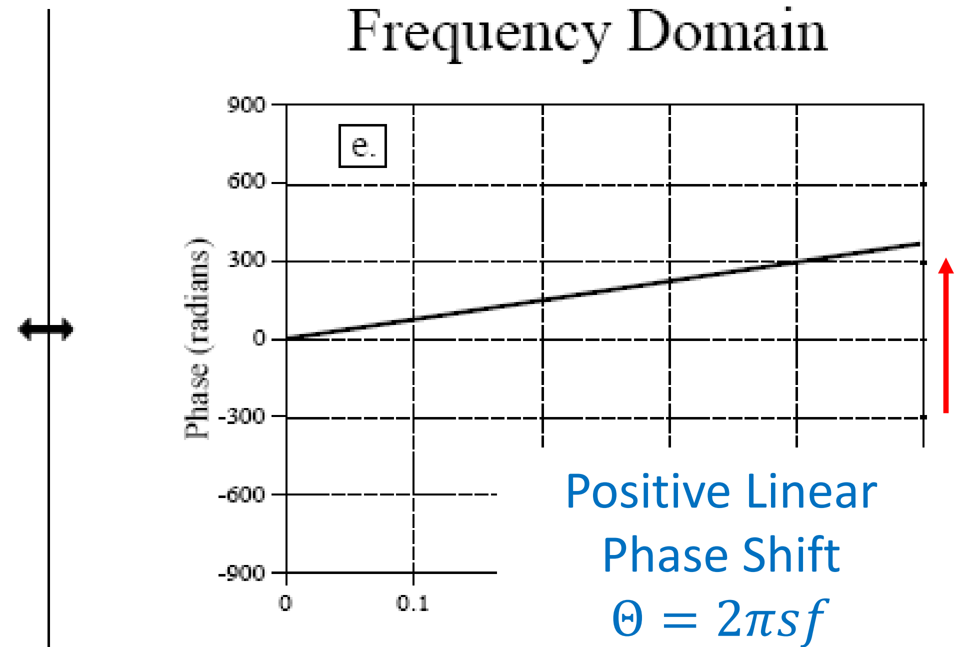
- At the 0 time reference the phase is zero
- Shift left – Positive phase shift
- Shift right – Negative phase shift



Time Shift Illustration

- Time shift left $[n+s]$-- Positive phase shift

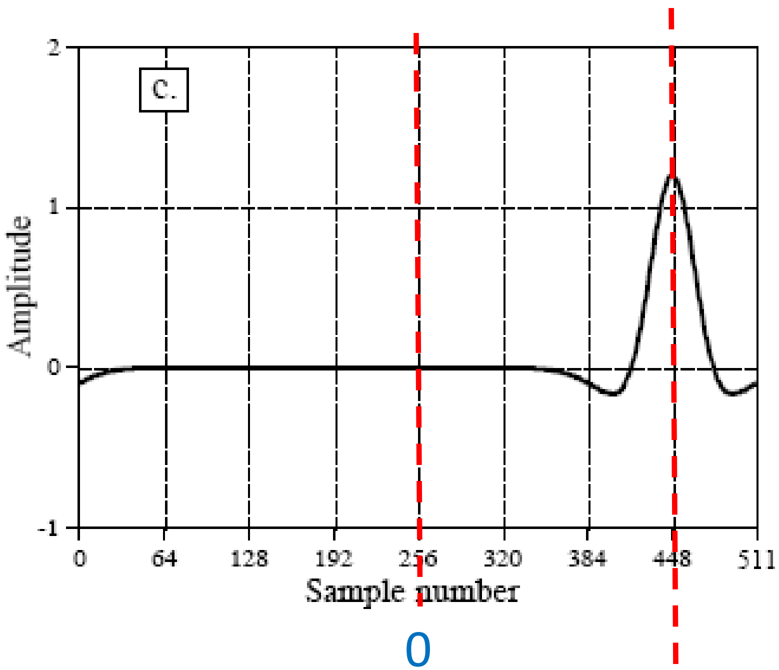




Time Shift Illustration

- Time shift right $[n - s]$-- Negative phase shift

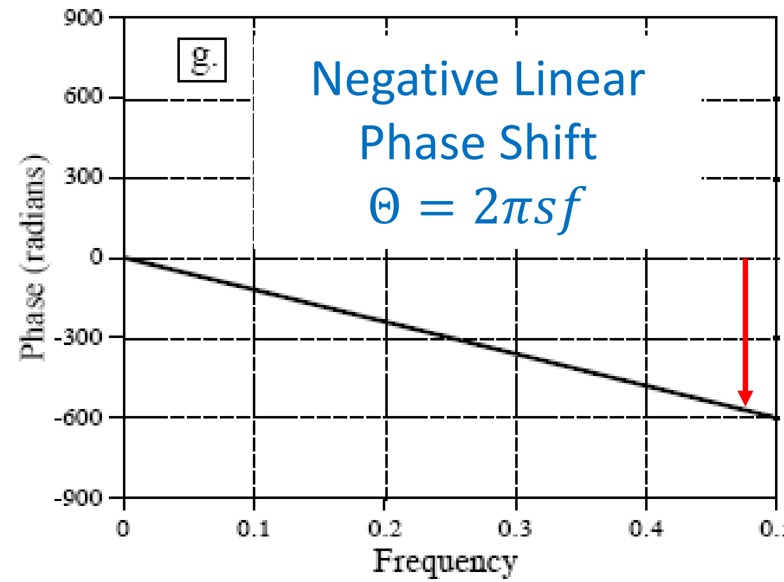
Time Domain



0

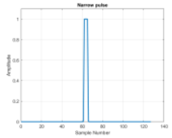
$x[n - s]$ Shift

Frequency Domain



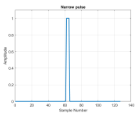
Time Compression and Expansion

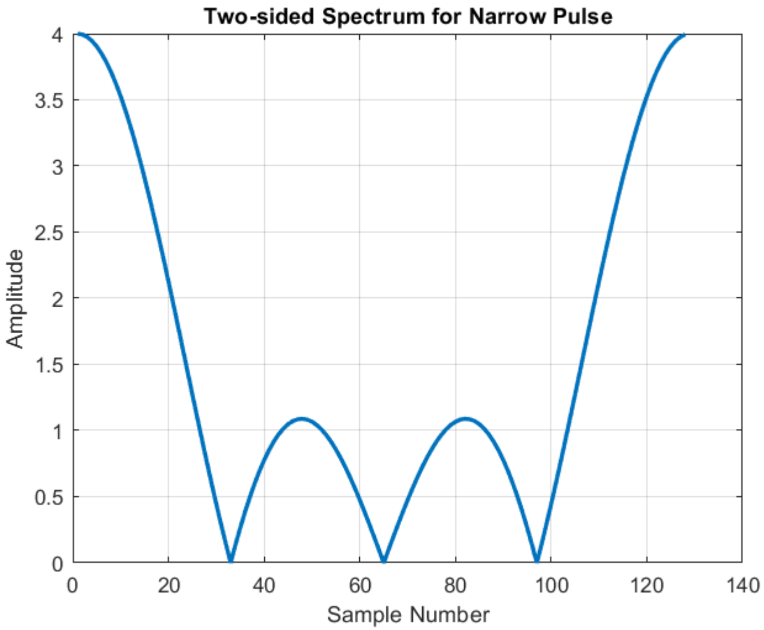
- What happens to the frequency domain if we compress or expand a signal in the time domain
- Assume we have a narrow pulse that is 4 samples long



Time Compression and Expansion

- Find the two-sided DFT of the pulse
- What happens if I make the pulse wider?

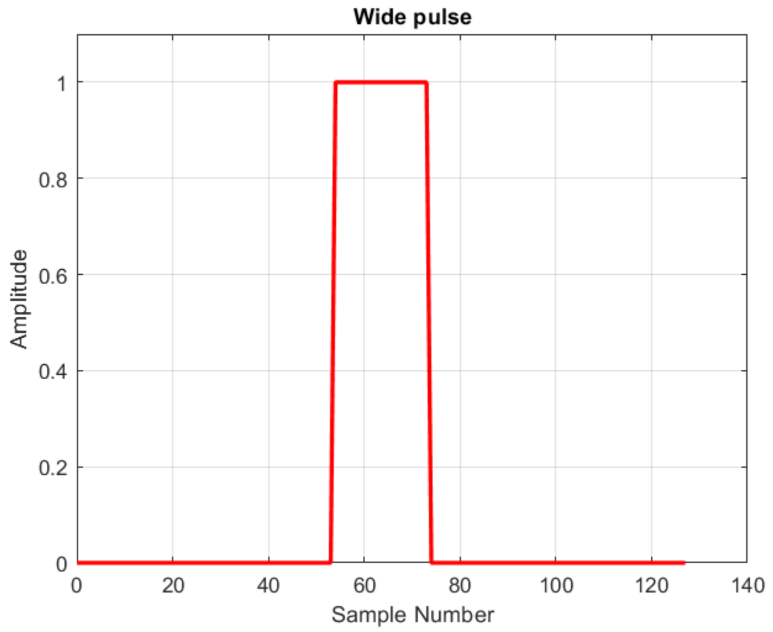


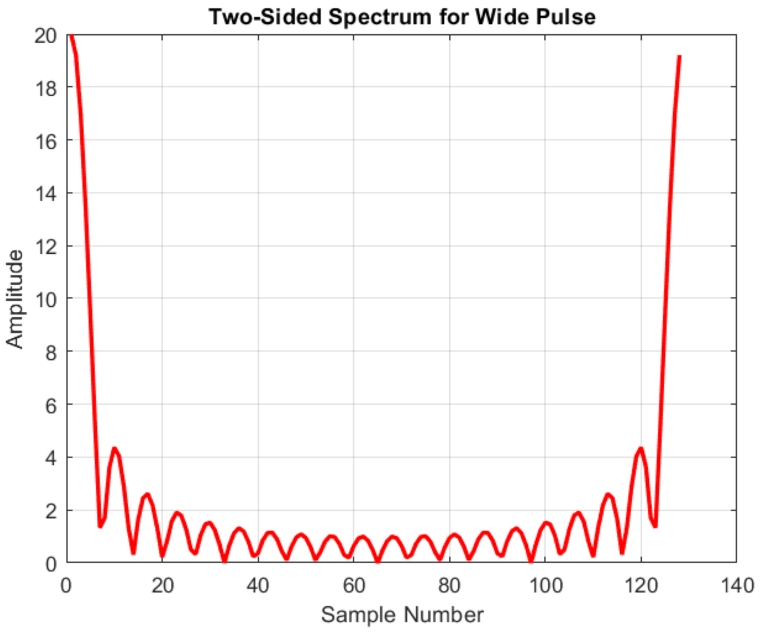


- The magnitude of the DFT of a pulse is a SINC function in the frequency domain

Expand the Pulse Width

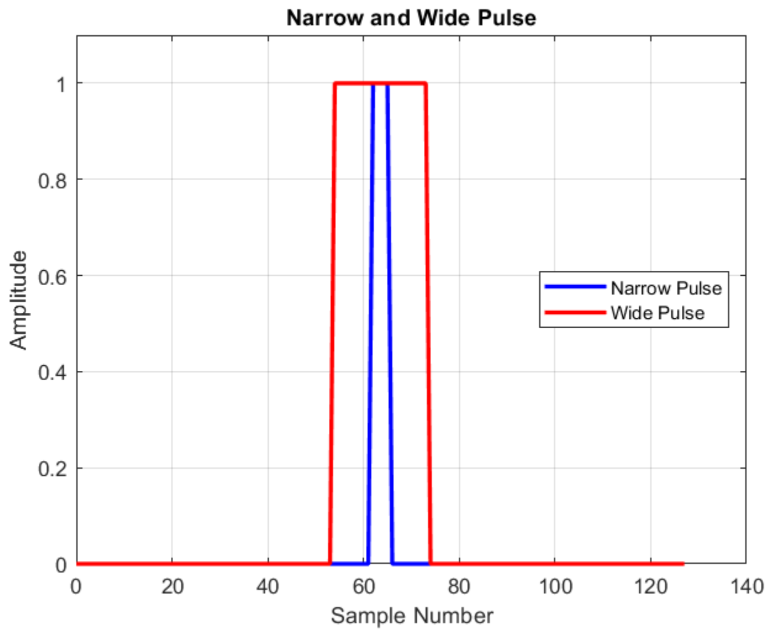
- Making the pulse wider narrows the spectrum
- It has fewer high frequency components

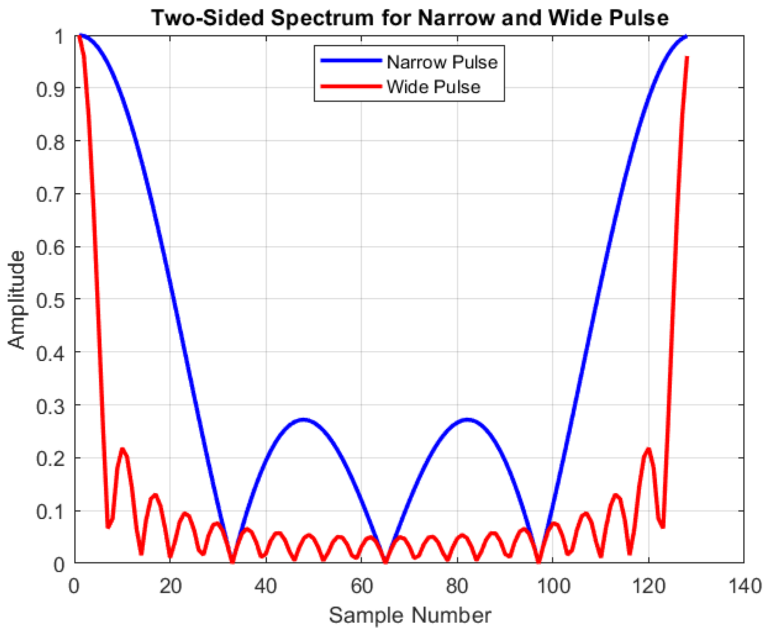




Comparing Pulse Widths

- Signals that change faster in the time domain have more high frequency content than signals that change more slowly





DFT Properties Summary

- Time domain convolution performed in the frequency domain
- Circular Convolution
- Time domain multiplication and the frequency domain
- Linear properties of the Fourier Transform
- Time shift and Time compression and their impact on the frequency domain