The Real Discrete Fourier Transform

Today's Topics

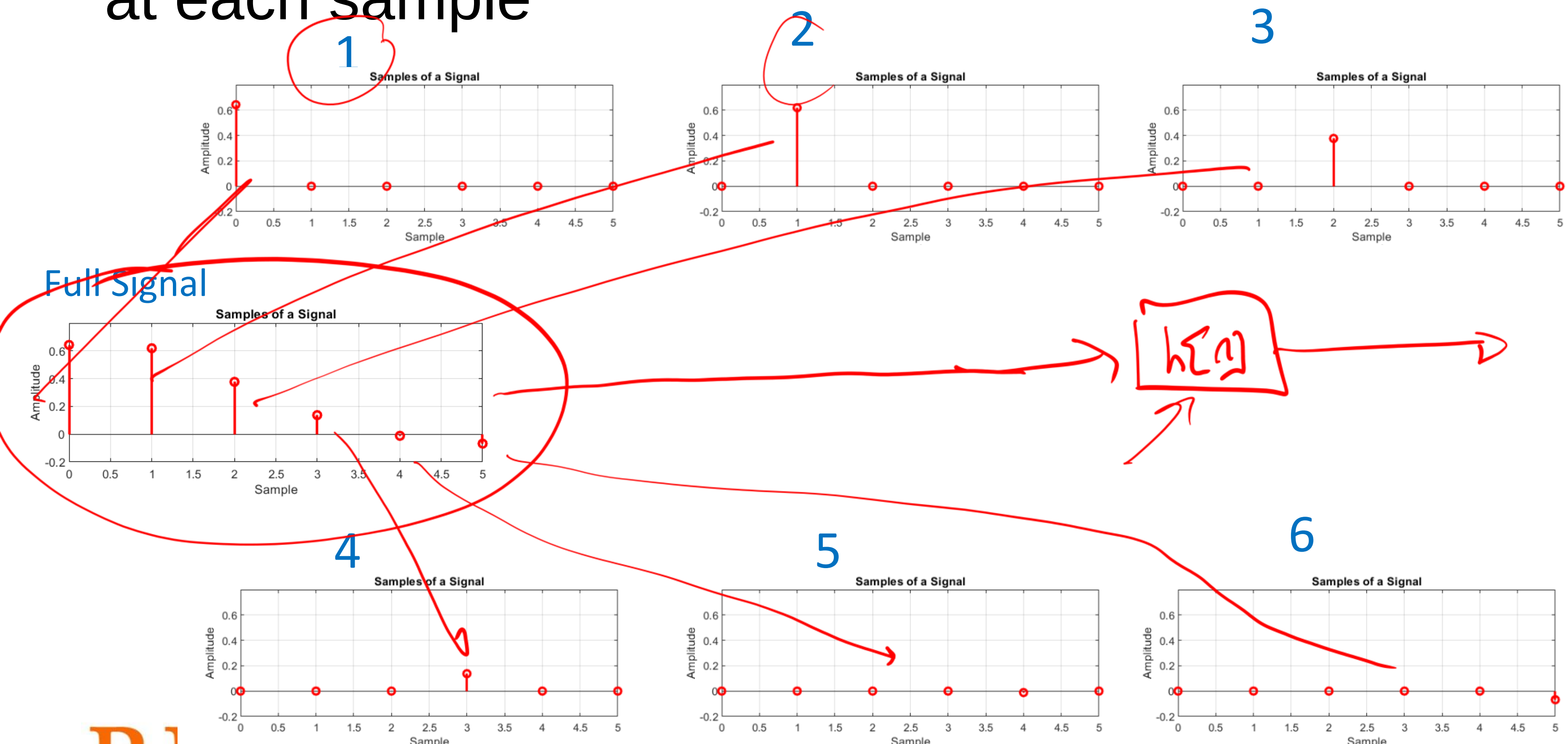
- The Real Discrete Fourier Transform
  - Decomposition
  - Synthesis
- Discussion of the different types of signals and Fourier Transforms
- The Real DFT - Specifics
  - Samples
  - Basis functions
  - Synthesis equation
  - Scaling

Decomposition of Signals

- In discussion of linear systems we discussed decomposing a signal into various parts
  - Impulse Decomposition – Breaking into impulses
    - Step Decomposition
  - Even and Odd Function Decomposition
  - Interlaced Decomposition
  - Fourier Decomposition -- Our focus for today

Impulse Decomposition -- Review

- What if we decompose the signal into impulses at each sample

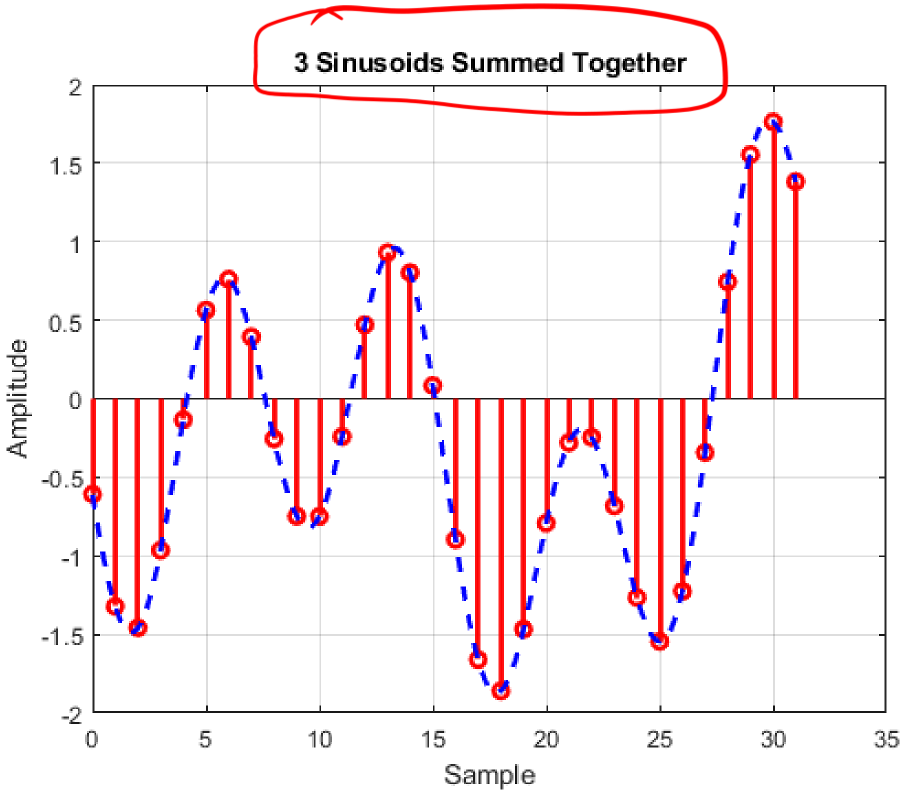


Impulse Decomposition Review

- We used impulse decomposition extensively in convolution
- Decomposed the input signal and applied the system impulse response, then combined

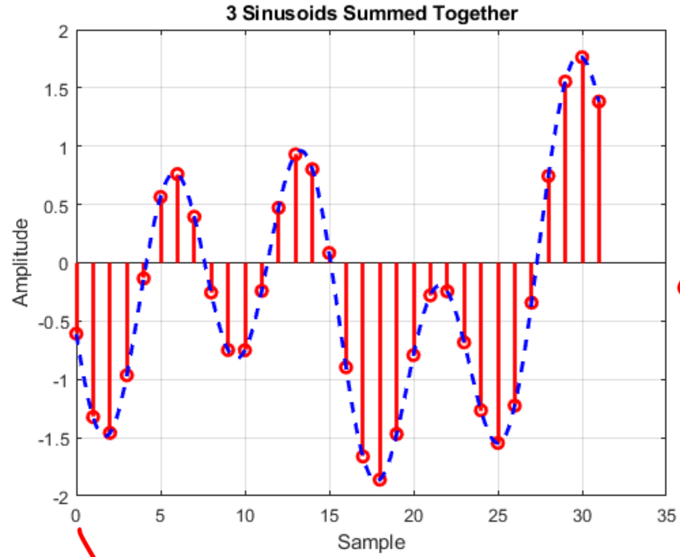
Fourier Decomposition

- Decompose the signal into a set of COSINE and SINE waves at different frequencies



3 Sinusoids added together

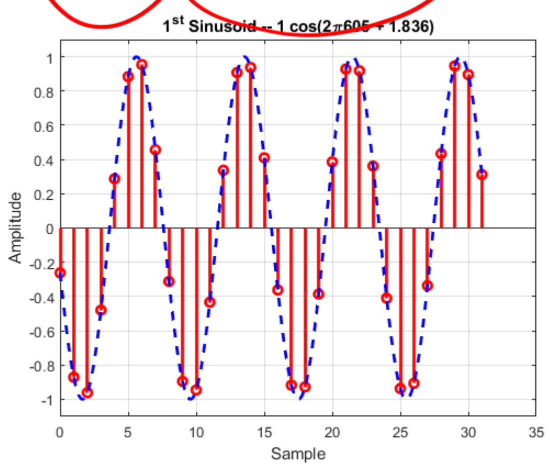
3 Sinusoids added together

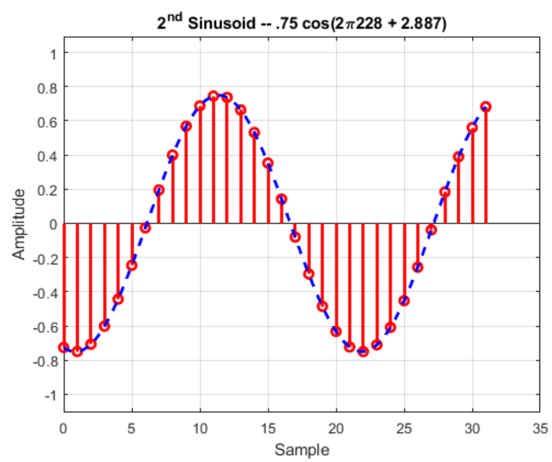


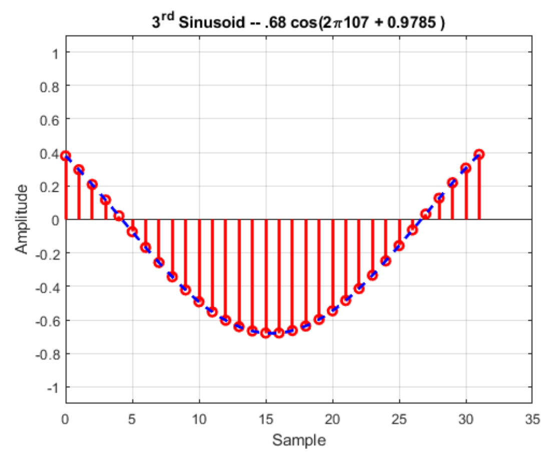
$605Hz, \phi = 1.836\ M = 1$

$228Hz, \phi = 2.89\ M = .75$

$107Hz, \phi = 0.98\ M = .68$







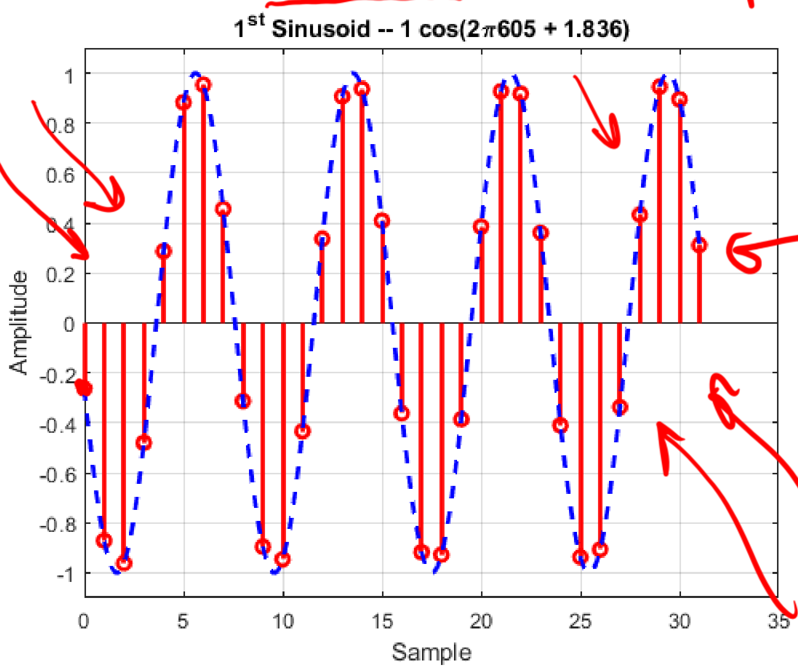
Decompose Each Sinusoid
First Sinusoid

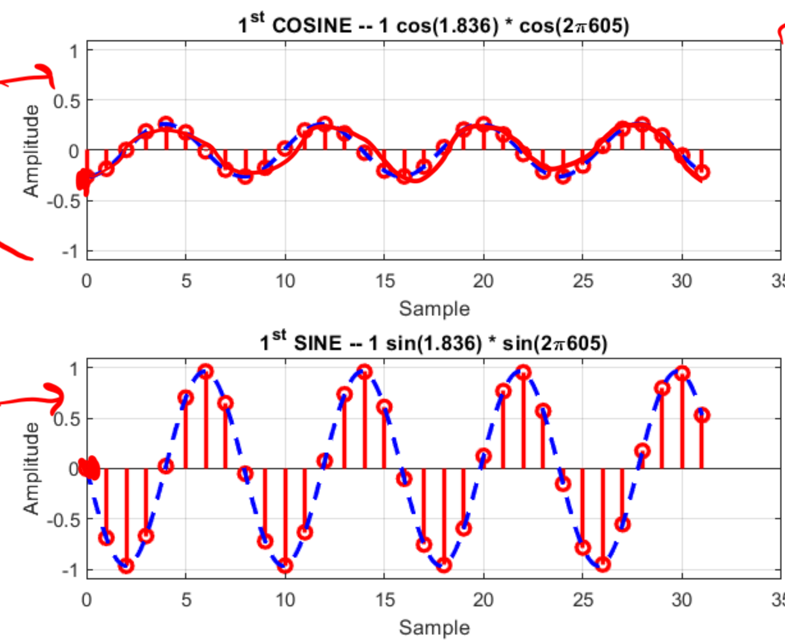
- Each sinusoid with a phase angle can be broken into a COS and SINE term

$$\cos(\omega t + \theta) = \cos(\theta)\cos(\omega t) - \sin(\theta)\sin(\omega t)$$

2π(605) 2π(605) 2π(605)

$605Hz, \phi = 1.836\ M = 1$



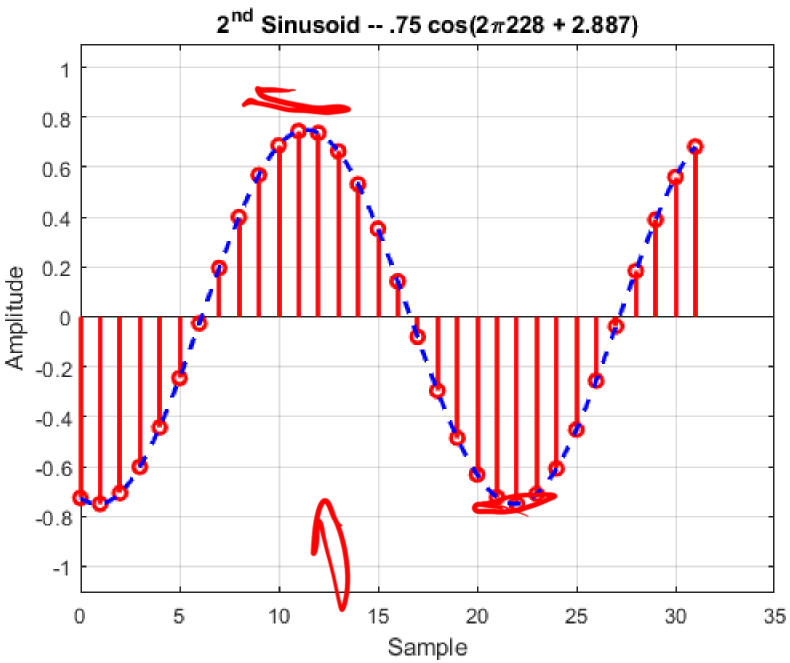


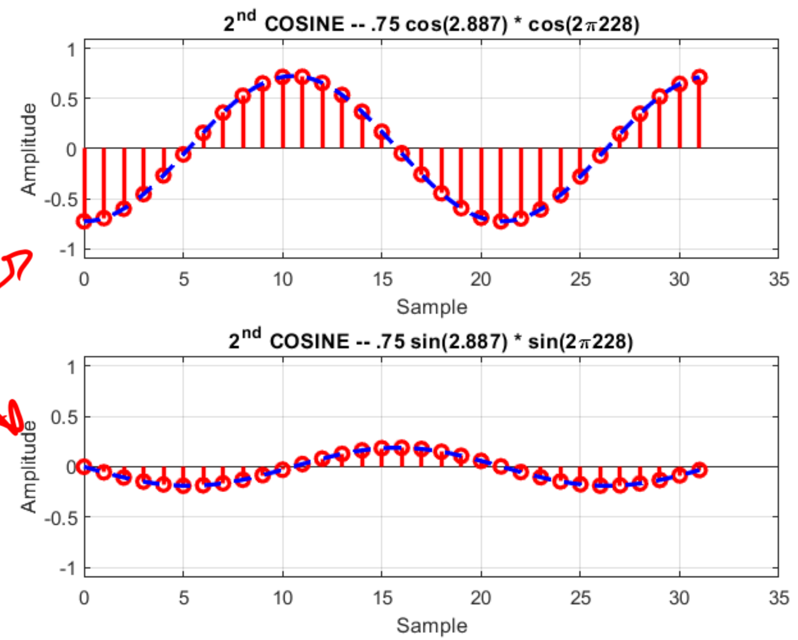
Second Sinusoid

$$cos(\omega t + \theta) = \sin(\theta)\cos(\omega t) + \cos(\theta)\sin(\omega t)$$

$2\pi(228)$ $2\pi(228)$

$228Hz, \phi = 2.89\ M = .75$



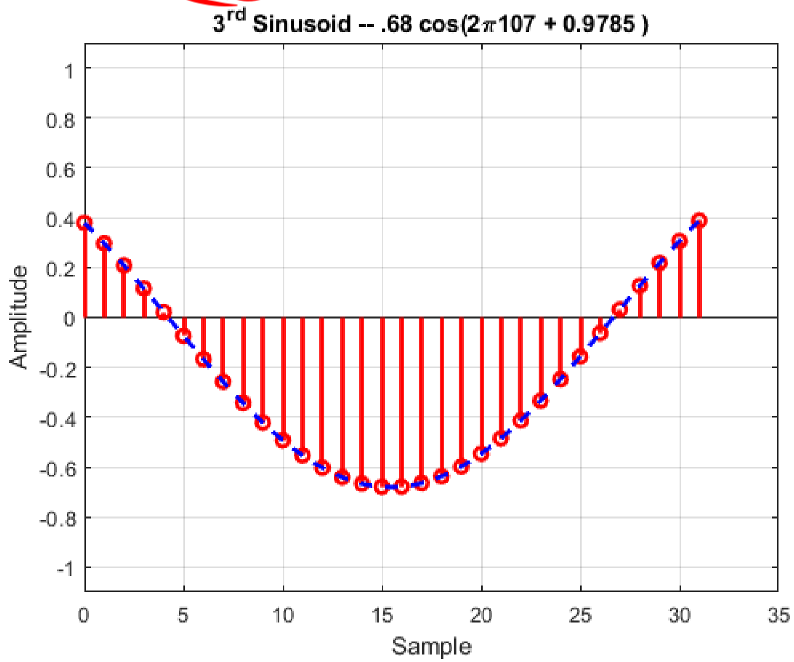


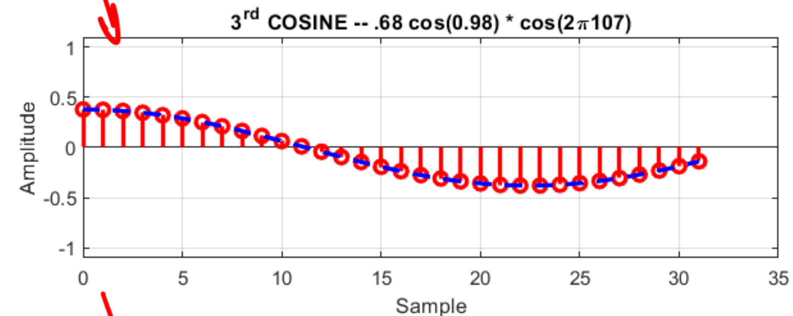
Third Sinusoid

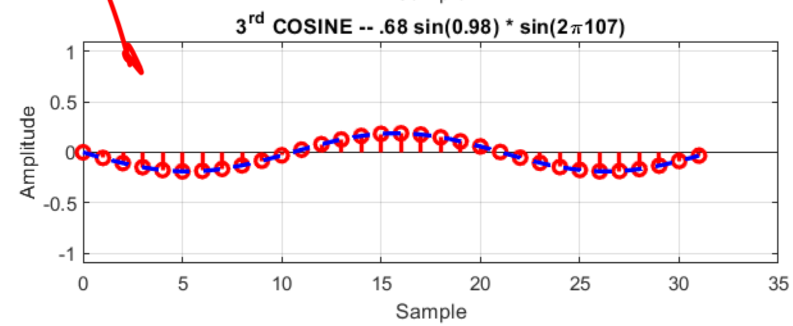
$$cos(\omega t + \theta) = \sin(\theta)\cos(\omega t) + \cos(\theta)\sin(\omega t)$$

2π(107)

$107Hz, \phi = 0.98\ M = .68$



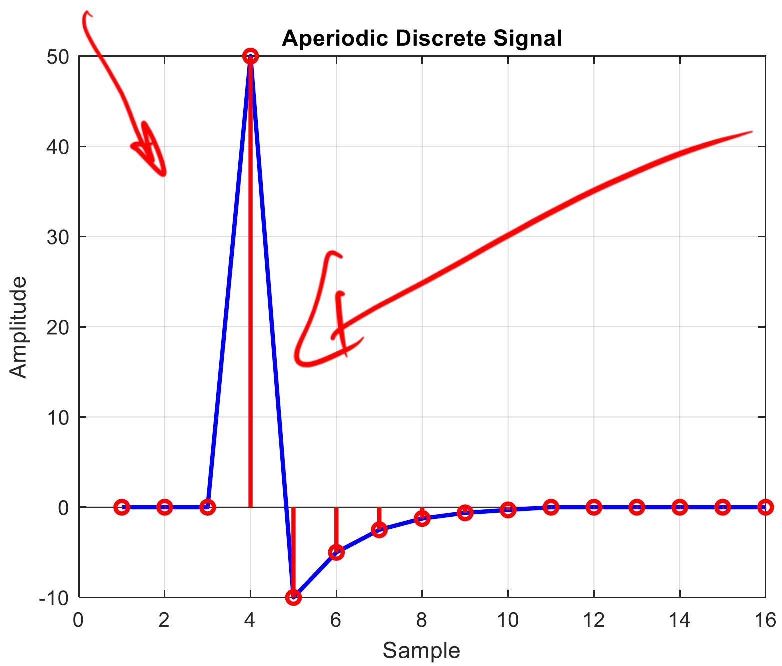




More Complex Signals

- We can easily see that a signal made up of sinusoids can be decomposed into SINE and COSINE terms

- Can I decompose this signal into COSINE and SINE signals?

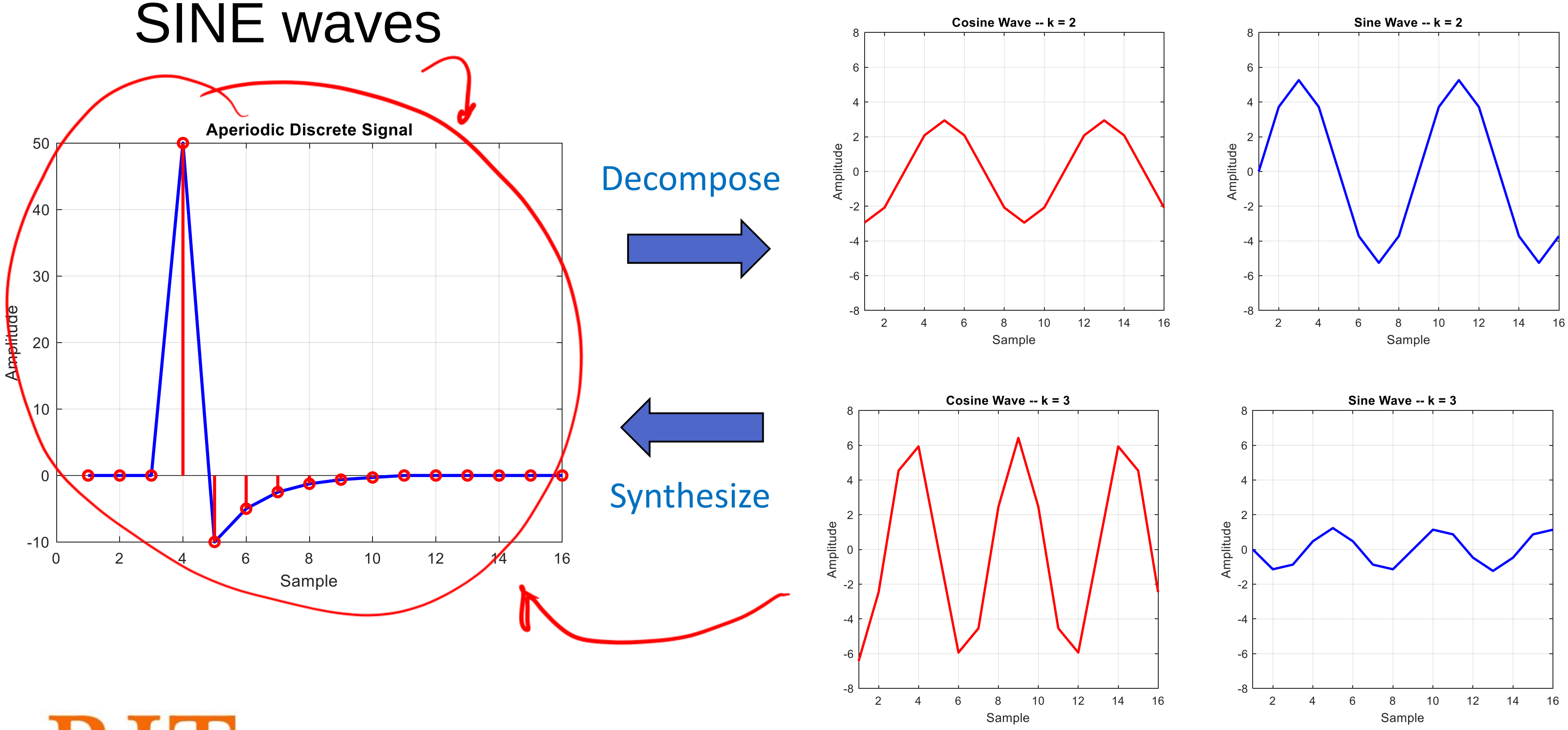


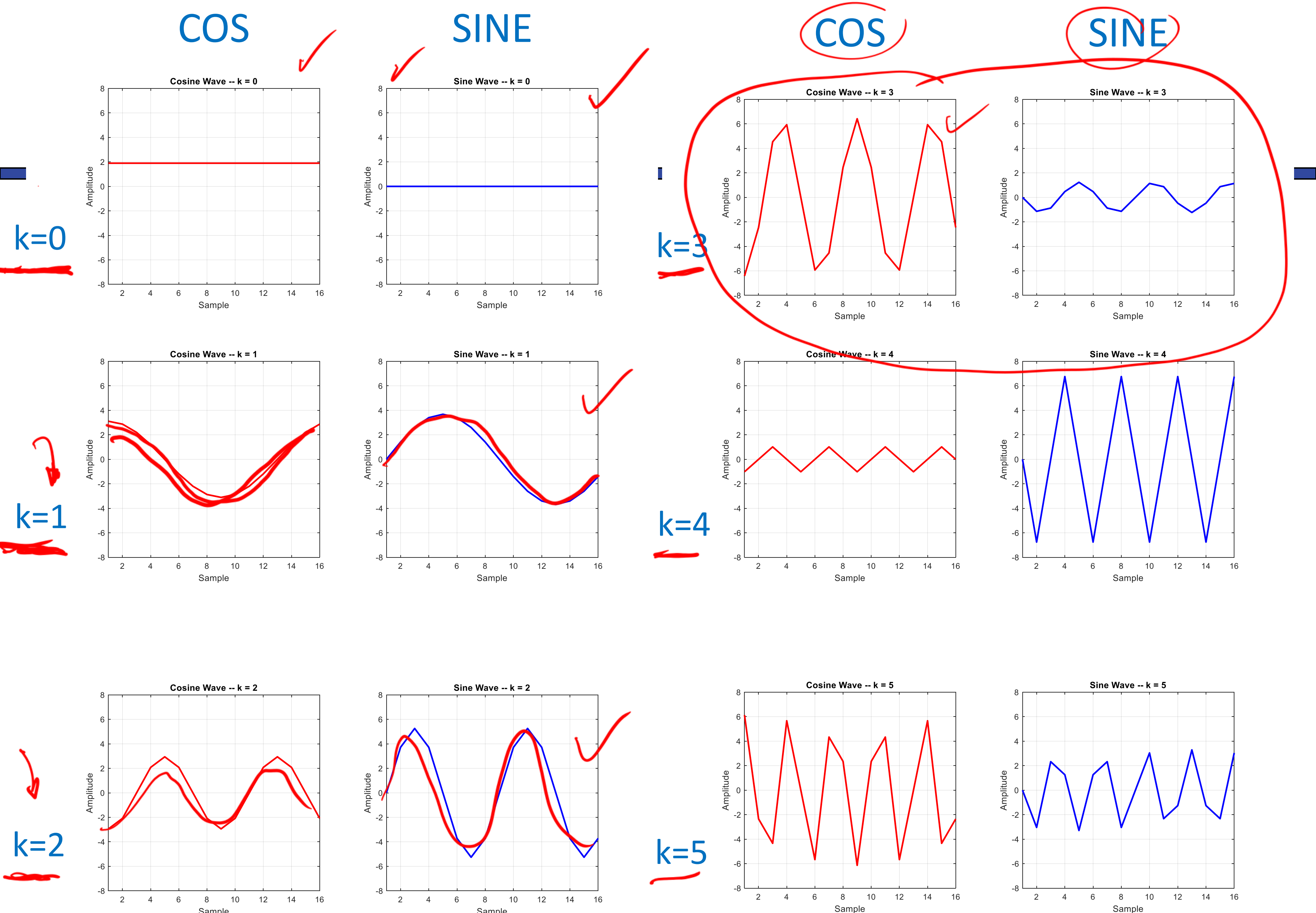
The Fourier Transform

- The Fourier Transform decomposes a signal into a set of sinusoidal signals.
- The Real Fourier Transform uses real numbers, as opposed to complex numbers
  - The complex sinusoids are broken down into the COS and SINE components

Decomposition and Synthesis

- Decompose – Break a signal into COS and SINE waves
- Synthesis – Reconstruct the signal from the COS and SINE waves



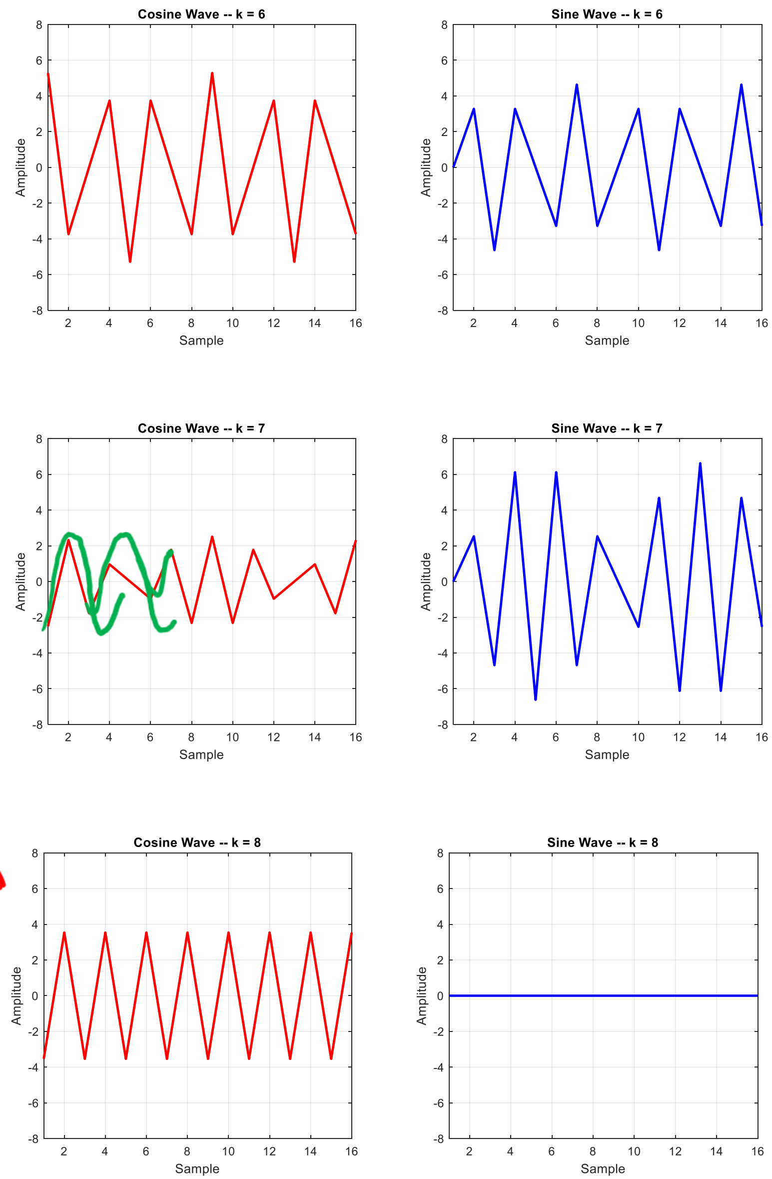

COS
SINE
COS
SINE
k=0
k=1
k=2
k=3
k=4
k=5
Cosine Wave -- k = 0
Sine Wave -- k = 0
Cosine Wave -- k = 1
Sine Wave -- k = 1
Cosine Wave -- k = 2
Sine Wave -- k = 2
Cosine Wave -- k = 3
Sine Wave -- k = 3
Cosine Wave -- k = 4
Sine Wave -- k = 4
Cosine Wave -- k = 5
Sine Wave -- k = 5
Amplitude
Sample

COS SINE

k=6

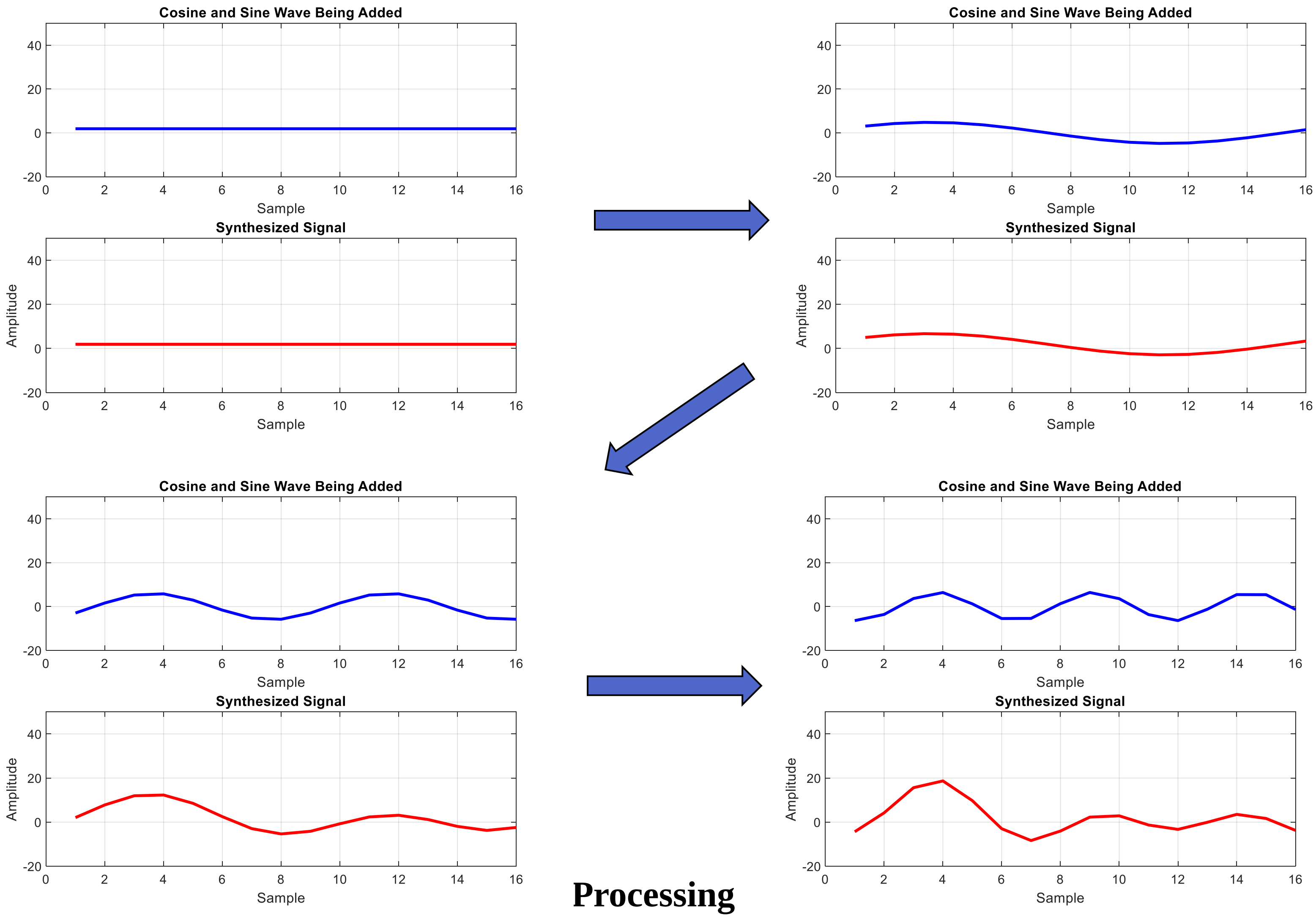
k=7

k=8



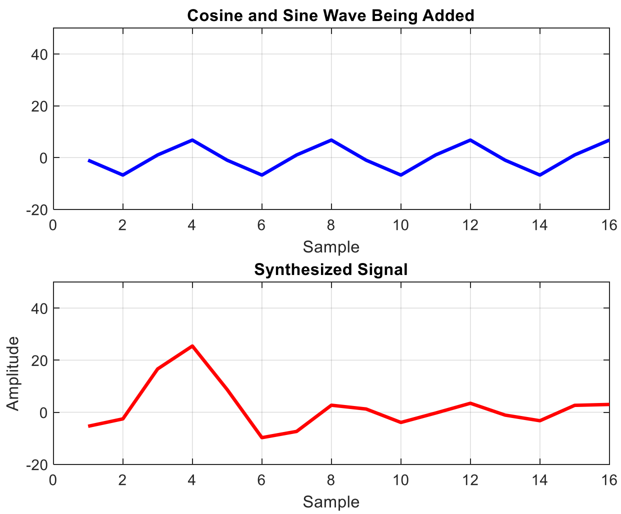
MATLAB Demo
DFT_Demo.m

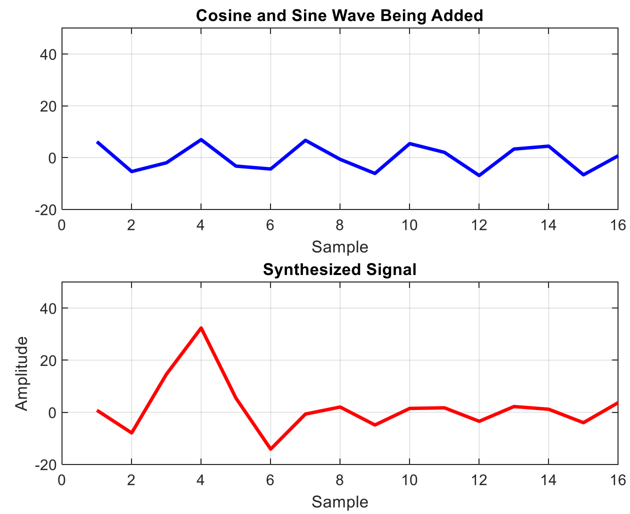
Can We Synthesize the Signal from the COS and SINE's?

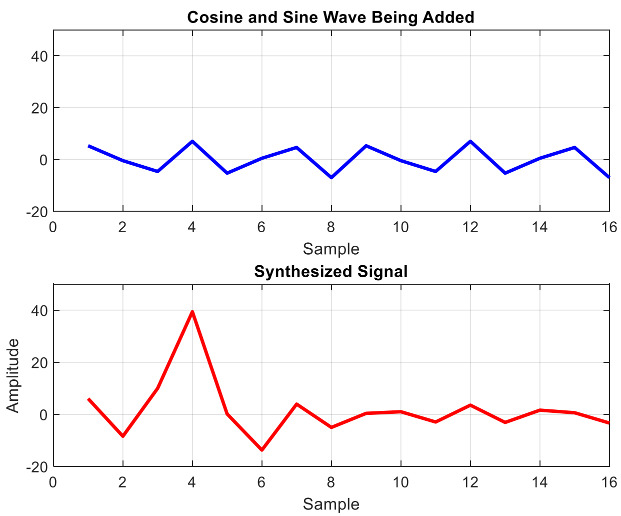


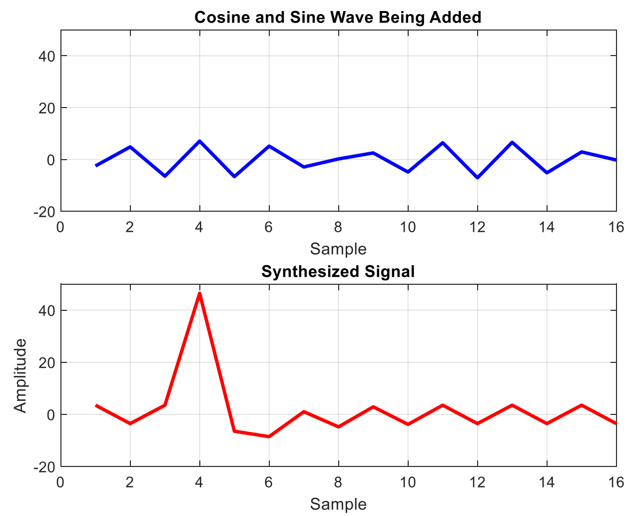
Processing

Can We Synthesize the Signal from the COS and SINE's?





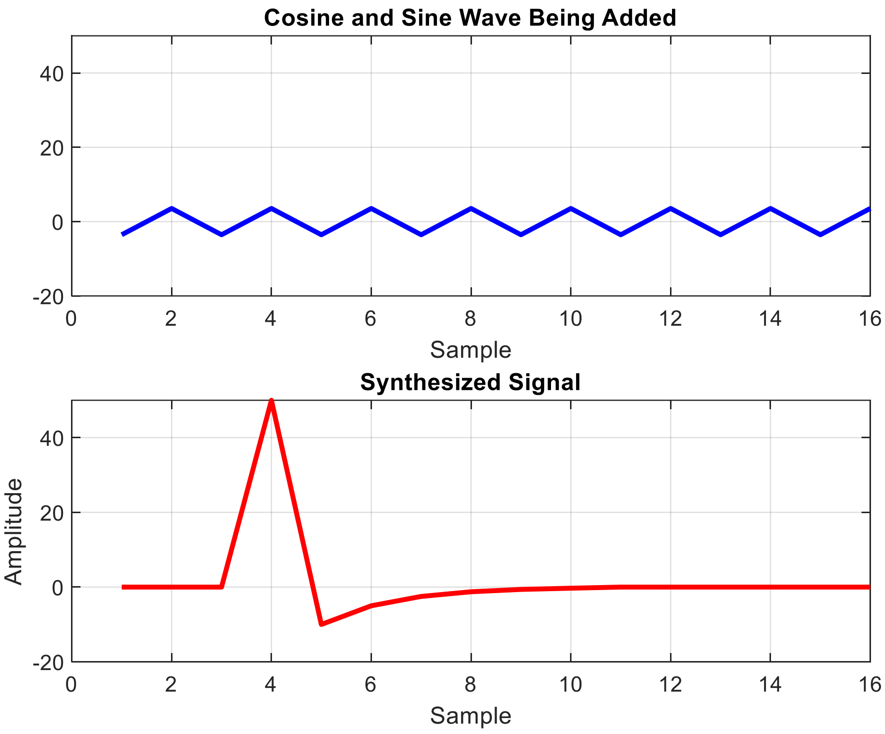




l Processing

The Original Signal Synthesized from Each COS and SINE

- The signal has been reproduced exactly from the 9 COS/SINE signal pairs

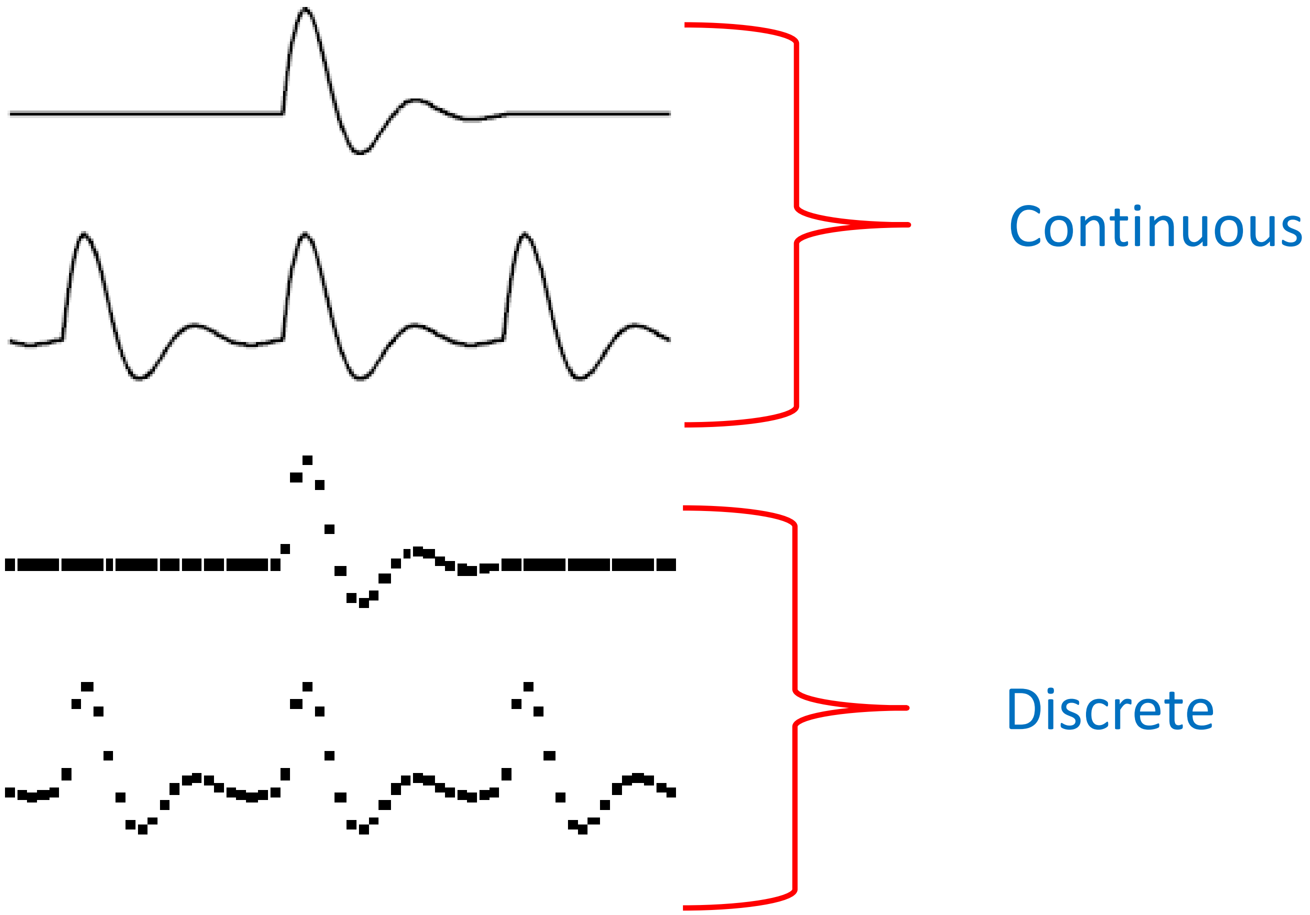


How do we decompose a signal?

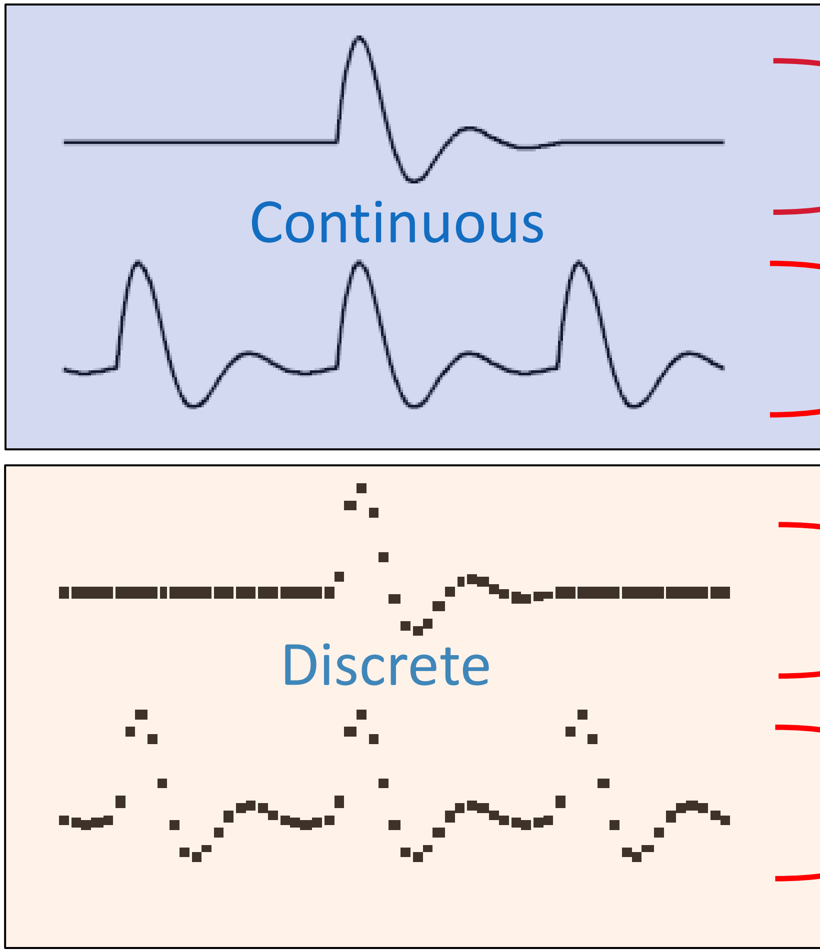
- A *<u>transform</u>* is used to decompose the signal and break it down into the COS and SINE components

- Which *<u>transform</u>* is used depends on the type of signal

Characterizing Signals



Characterizing Signals



Signal Type	Transform Type
Continuous Aperiodic	Fourier Transform
Continuous Periodic	Fourier Series
Discrete Aperiodic	Discrete Time Fourier Transform
Discrete Periodic	Discrete Fourier Transform

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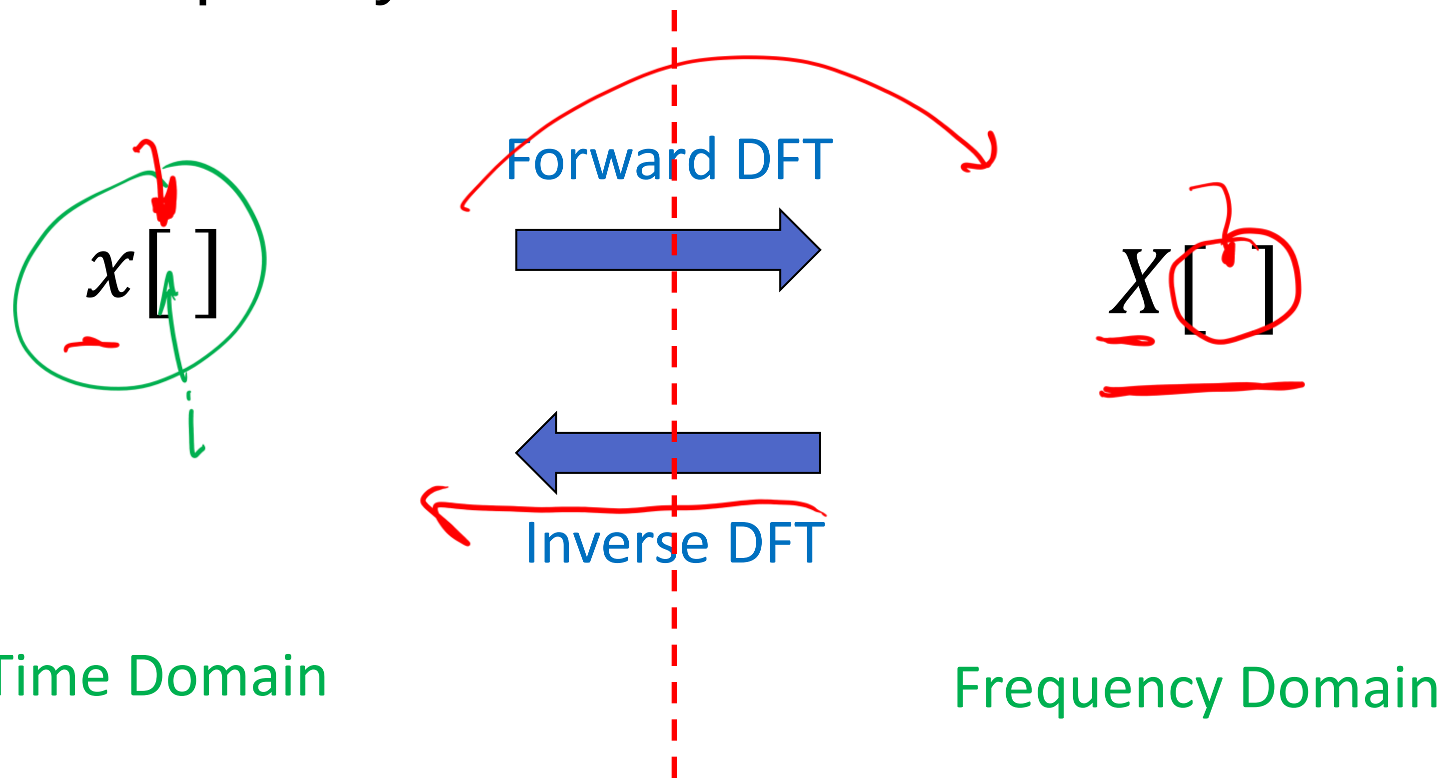
Discrete Fourier Transform

- We will be only be using the Discrete Fourier Transform (DFT)

- We will always be talking about discrete time samples of a signal that is assumed to be periodic.

Discrete Fourier Transform

- The DFT transforms a time domain signal into the frequency domain



N even

Discrete Fourier Transform

- N samples in the time domain produce $N/2 + 1$ samples in the frequency domain

k

Forward DFT

$x[\,]$ → $X[\,]$

N-Samples in the Time Domain

Inverse DFT

N/2 +1 Samples In the Frequency Domain

Time Domain

Frequency Domain

Discrete Fourier Transform

- The real part of the frequency domain signal are the COSINE amplitudes. Imaginary part are the SINE amplitudes

Frequency Domain

N/2 +1 Samples In the Frequency Domain

$$X[\ ]$$

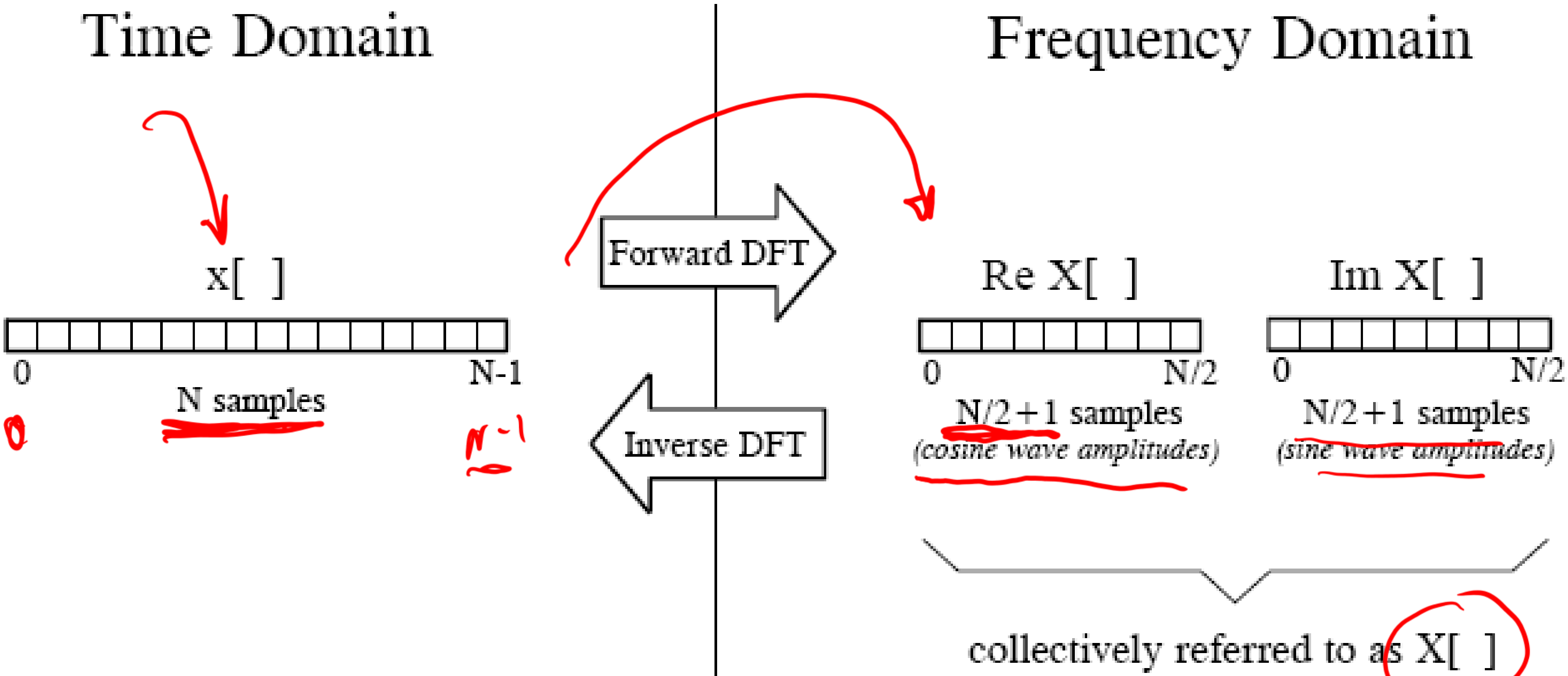
$$Re[X] \qquad\qquad Im[X]$$

Amplitude of the COSINE waves

Amplitudes of the SINE waves

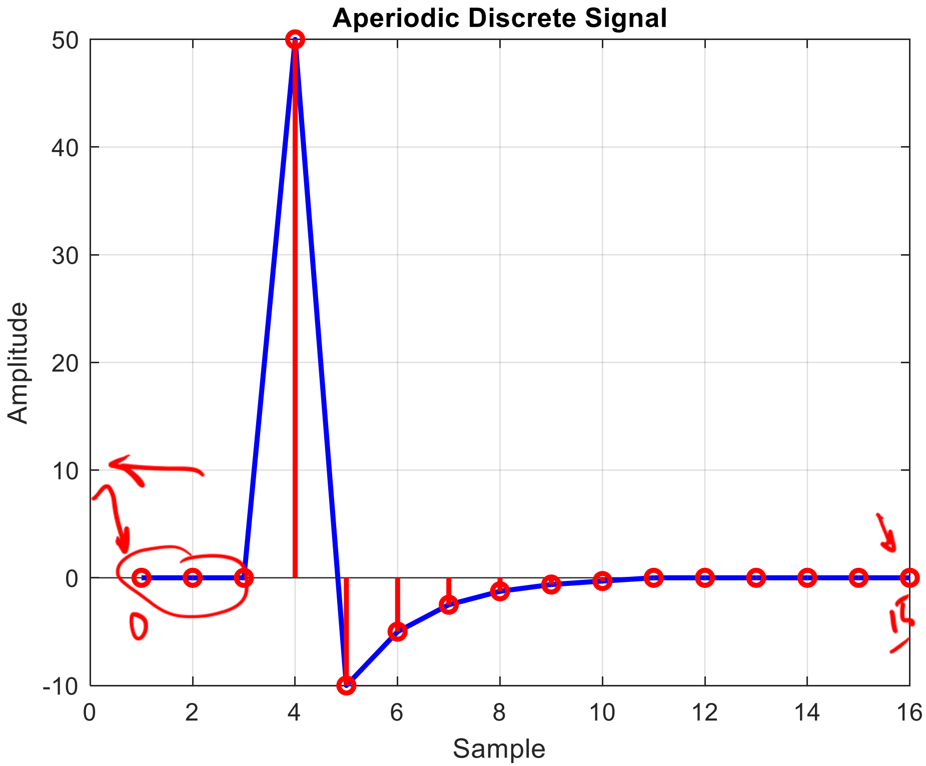
Real DFT: Time to Frequency Domain Transform

- Frequency Domain refers to the amplitude of cosines/sines



DFT of Our Previous Example

- The signal has N=16 samples in the time domain

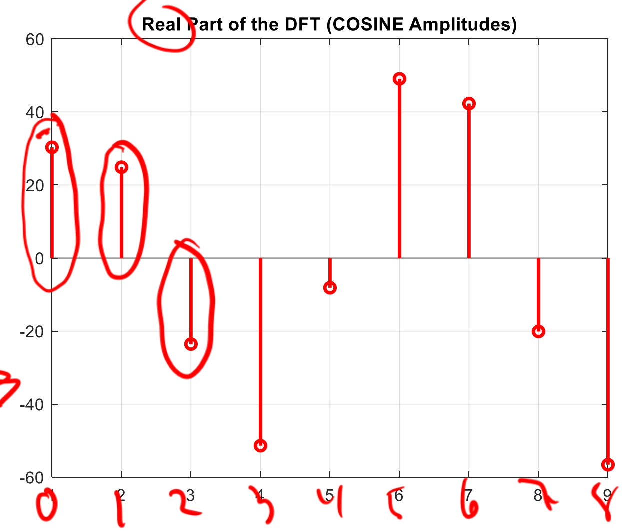


n	x[n]
0	0
1	0
2	0
3	50
4	-10
5	-5
6	-2.5
7	-1.25
8	-0.625
9	-0.3
10	0
11	0
12	0
13	0
14	0
15	0

Forward DFT Results

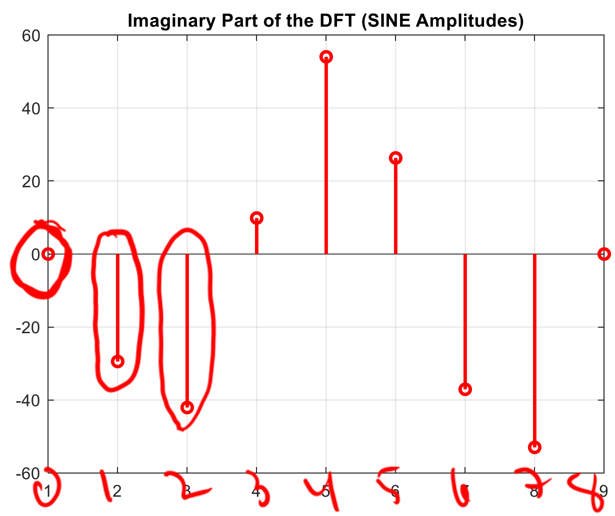
N/2+1 Samples
9 Samples
COSINE Amplitudes

n	Re[X]
0	30.33
1	24.87
2	-23.54
3	-51.36
4	-8.13
5	49.08
6	42.29
7	-20.09
8	-56.58



N/2+1 Samples
9 Samples
SINE Amplitudes

n	Im[X]
0	0.00
1	-29.44
2	-42.06
3	9.87
4	54.05
5	26.33
6	-37.06
7	-52.98
8	0.00



Forward DFT Results

N/2+1 Samples
9 Samples
COSINE Amplitudes

n	Re[X]
0	30.33
1	24.87
2	-23.54
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4	-8.13
5	49.08
6	42.29
7	-20.09
8	-56.58

Scaled amplitude of the 1st (k=0) COSINE

Scaled amplitude of the 4th (k=3) COSINE

N/2+1 Samples
9 Samples
SINE Amplitudes

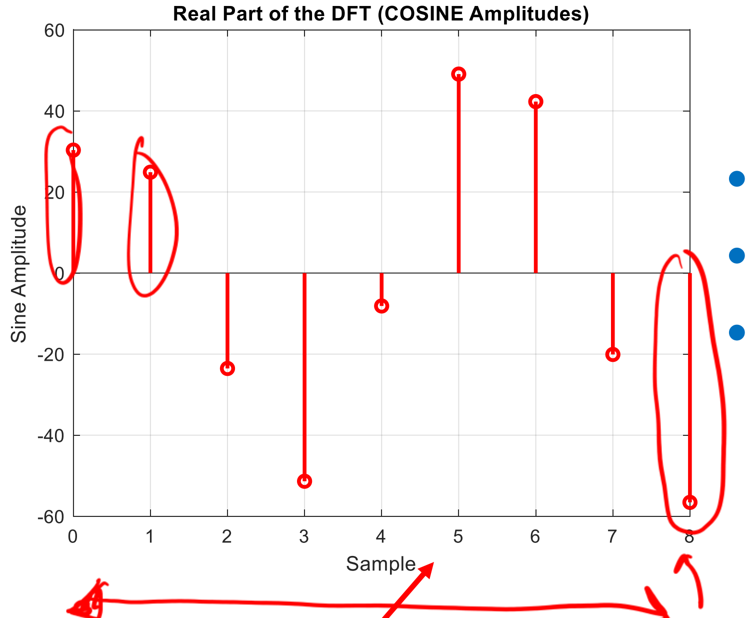
n	Im[X]
0	0.00
1	-29.44
2	-42.06
3	9.87
4	54.05
5	26.33
6	-37.06
7	-52.98
8	0.00

Scaled amplitude of the 3rd (k=2) SINE

Scaled amplitude of the 7th (k=6) SINE

Frequency Domain Independent Variable

- What is the independent variable in the frequency domain? 4 different representations



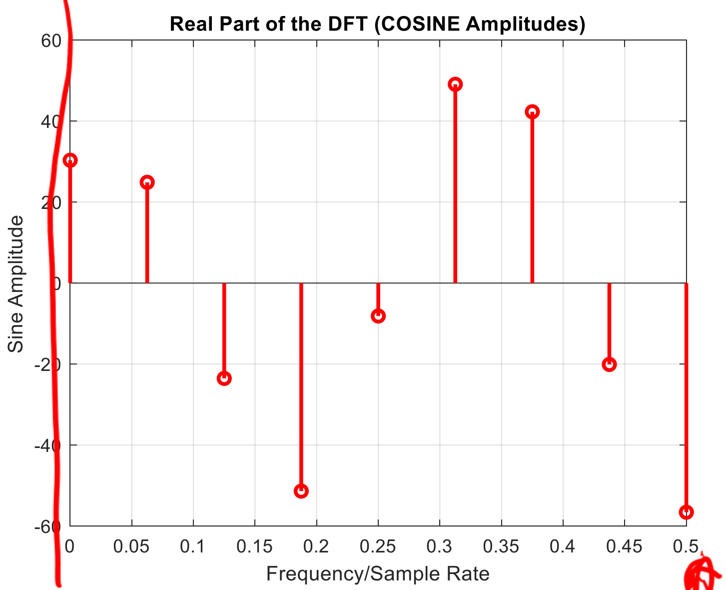
- Represents the 0 to N/2 samples
- An integer value
- Useful in programming (e.g. indexing)

What does this axis represent?

Sample Number

Frequency Domain Independent Variable

- Fraction of the sample rate



0
0.5

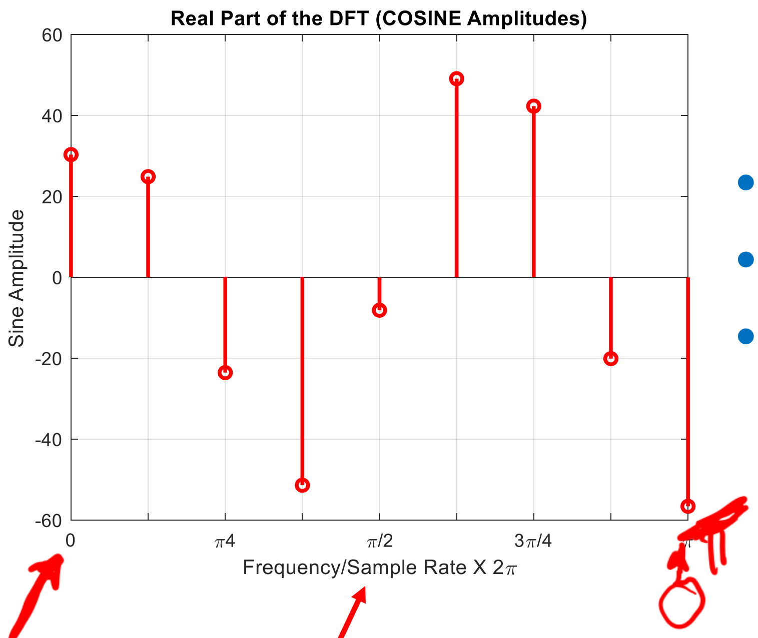
- Represents fraction of the sample rate
- Maximum of 0.5 – Nyquist Rate

What does this axis represent?

Fraction of Sampling Rate

Frequency Domain Independent Variable

- Natural frequency in rad/sec



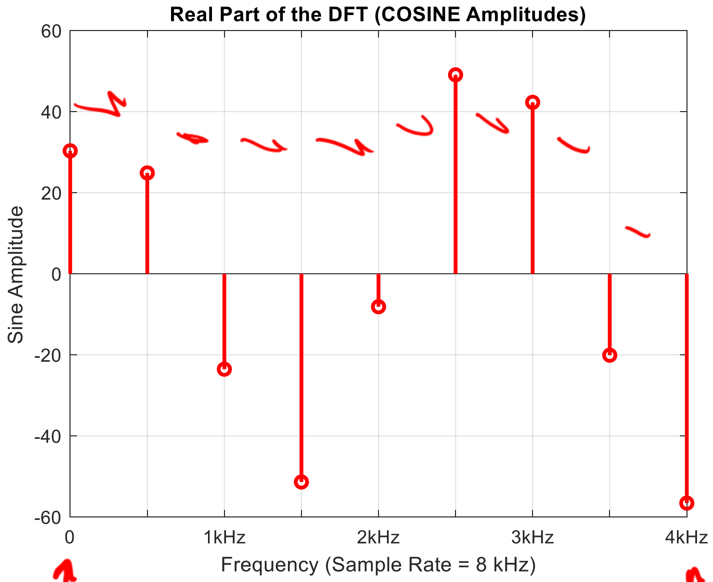
- Natural Frequency -- Radians
- Fraction of the sample rate times 2π
- Maximum of π – Nyquist Rate

What does this axis represent?

Fraction of Sampling Rate

Frequency Domain Independent Variable

- The absolute frequency



- Absolute Frequency
- Maximum of the Nyquist Rate
- Assume 8 kHz sample rate

0

4 KHz

What does this axis represent?

Fraction of Sampling Rate

What do the COS and SINE waves Represent

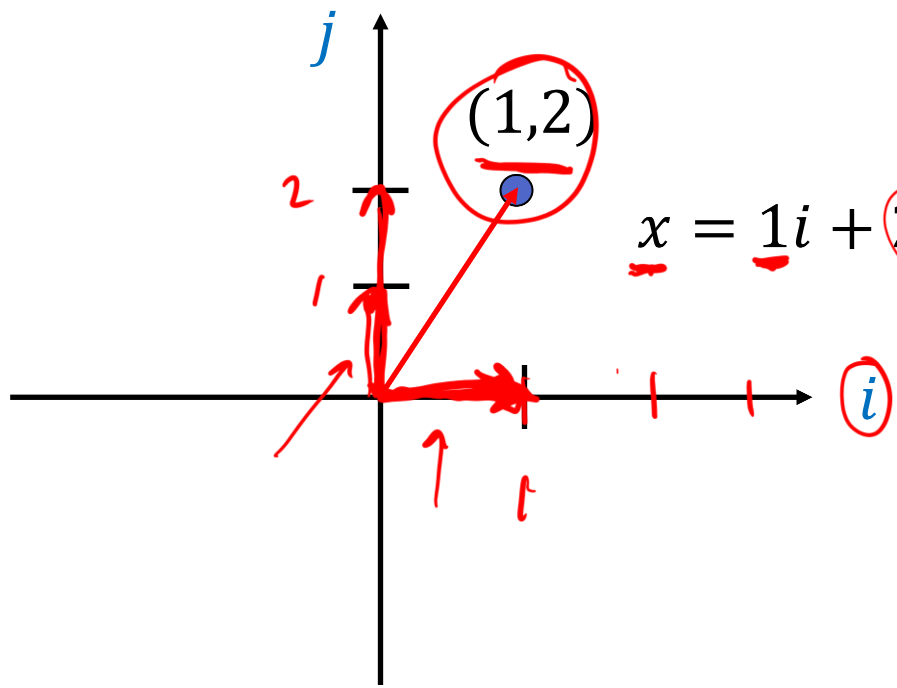
- The COSINE and SINE wave signals are BASIS functions

- What is a BASIS function?

  - A set of orthonormal functions that when linearly combined can create any function in the space
  - Orthonormal – Orthogonal and Unit Length functions

BASIS function example

- Consider the cartesian plane – Any point in the plane can be described by a linear combination of the BASIS functions $i = \binom{1}{0}$ $j = \binom{0}{1}$



$x = 1i + 2j$

BASIS Functions for the DFT

- The BASIS functions for the DFT are

$$c_k[i] = \cos(2\pi ki/N)$$

$$s_k[i] = \sin(2\pi ki/N)$$

- These represent COS and SINE functions that have a frequency of k/N

- The COS and SINE function will complete k cycles in N samples

BASIS Functions for the DFT

- The BASIS functions for the DFT are

$$c_k[i] = \cos(2\pi ki/N)$$

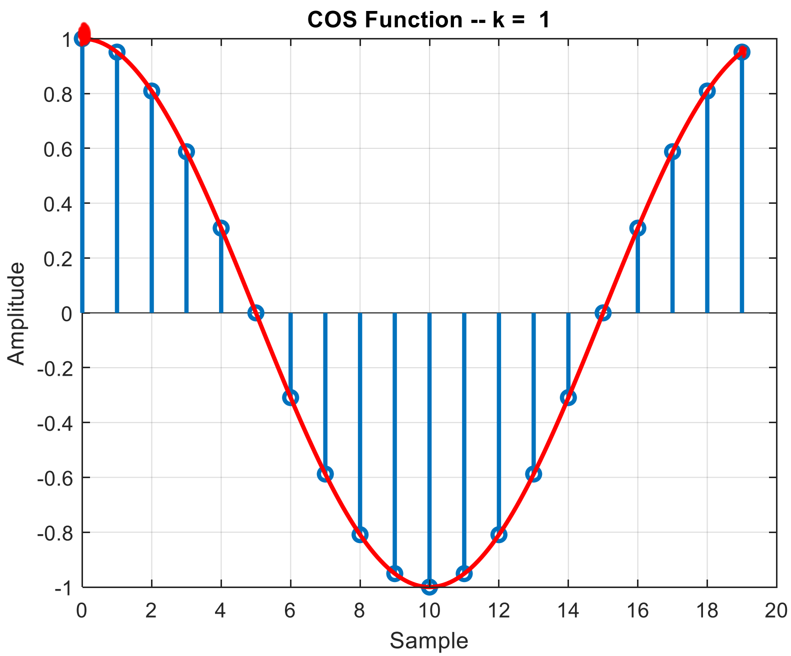
$$s_k[i] = \sin(2\pi ki/N)$$

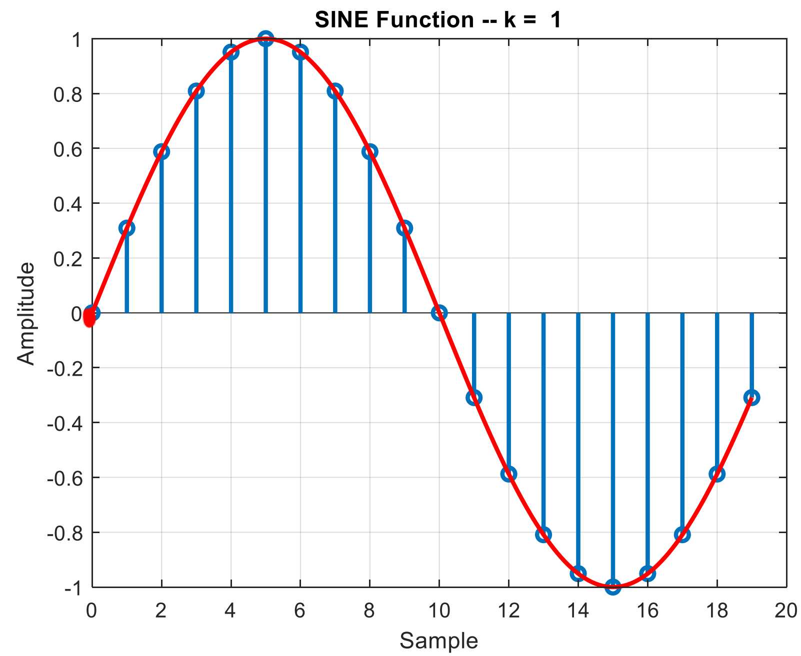
- i goes from 0 to N-1 and represents the time domain

- k goes from 0 to N/2 and represents the frequency

Example Basic Functions
k = 1, N = 20

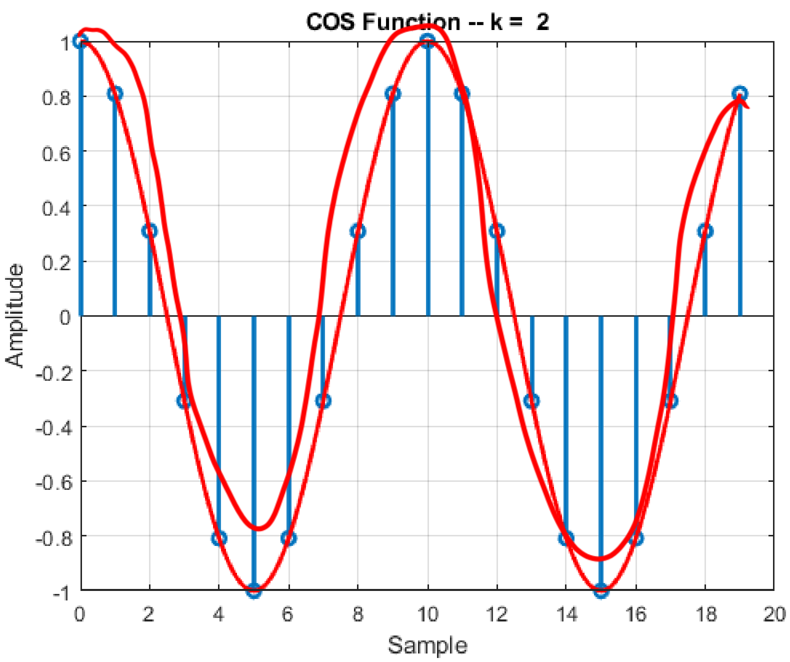
- The function completes 1 cycle in N samples

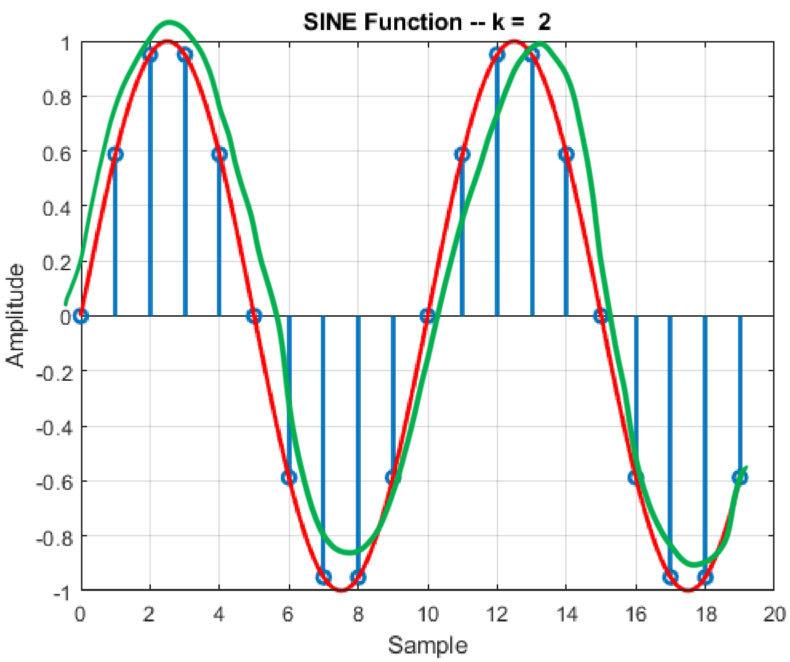




Example Basic Functions
k = 2, N = 20

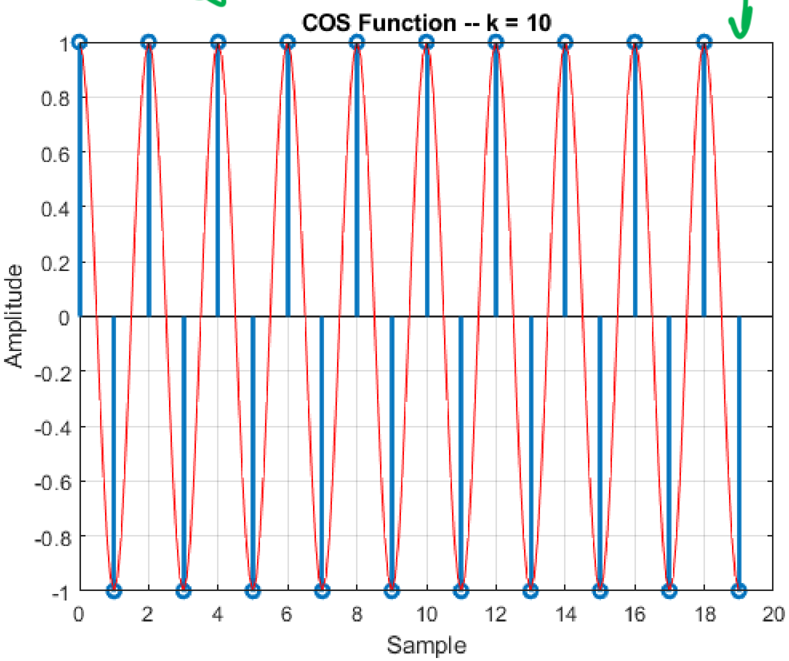
- The function completes 2 cycles in N samples

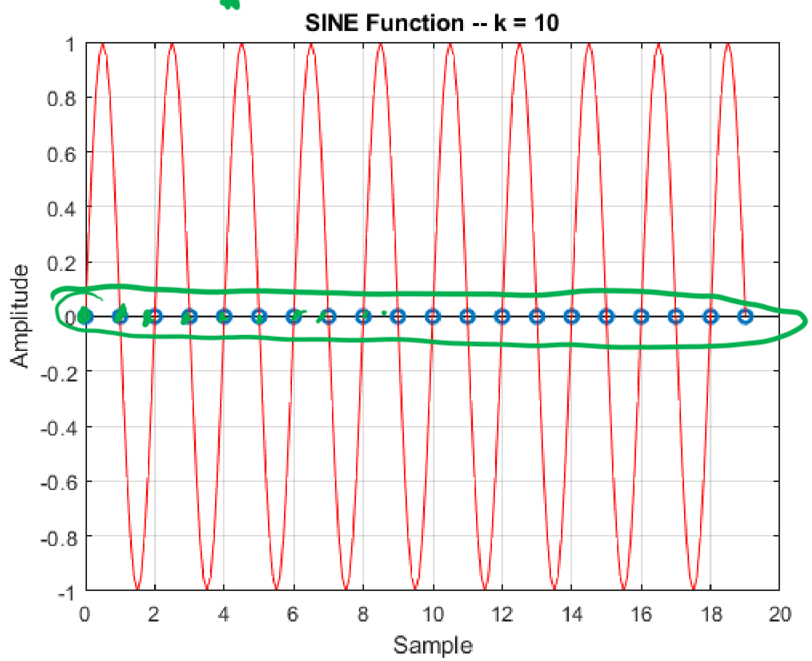




Example Basic Functions
k = N/2 = 10, N = 20

- The function completes 10 cycles in N samples



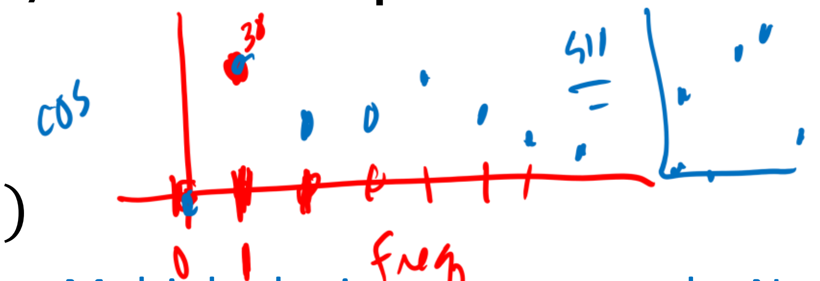


But How do We Get $X[k]$?

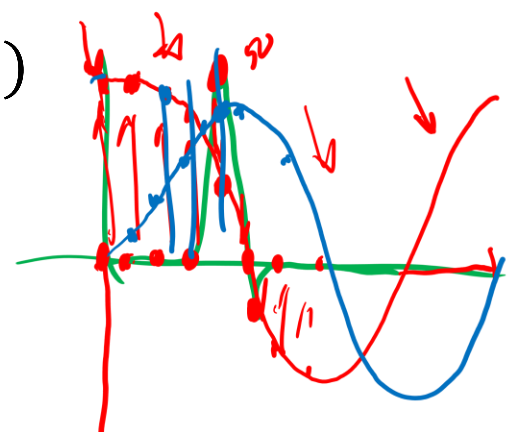
- We *correlate* the input sequence with each COS and SINE wave at $N/2+1$ frequencies

$$Re(X[k]) = \sum_{i=0}^{N-1} x[i]\cos(2\pi ki/N)$$

$$Im(X[k]) = \sum_{i=0}^{N-1} x[i]\sin(2\pi ki/N)$$



Multiply the input sequence by N samples of the cosine and sine signals for each frequency k



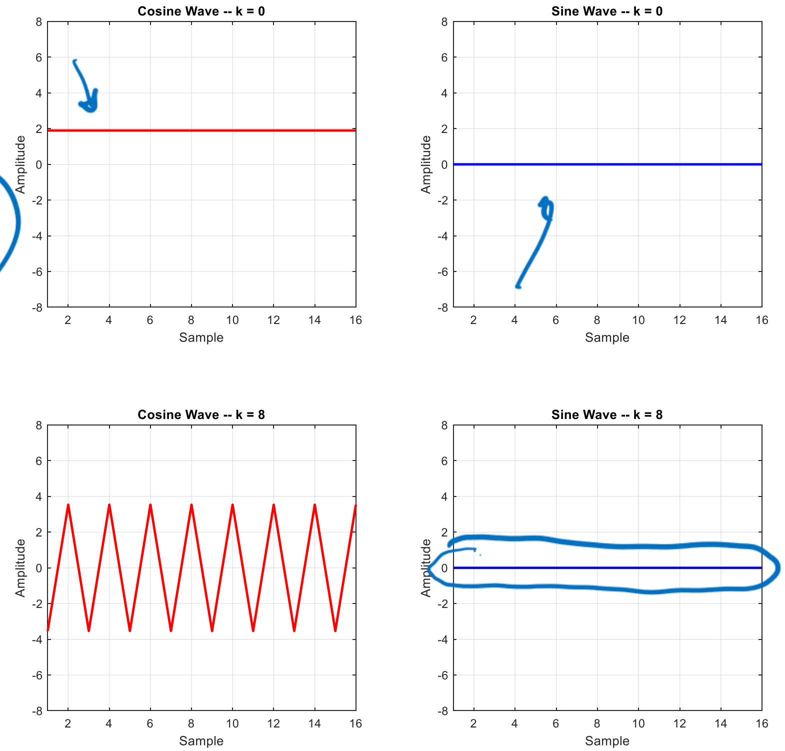
Basis Functions of the DFT

- Note that the coefficients of $Im\ X[0]$ and $Im\ X[N/2]$ are always zero.

COS SINE

$Re(X[0])$ is the DC component

k=0



Imaginary values are all zero

From previous example with N=16

N/2

k=8

Imaginary values are all zero

Computing the DFT

- Create the COS and SINE signals at k = 0
- Then multiply by each point of the input and sum

			k=0, N=16			k=0, N=16	
i	x[i]		COS(2*pi*k*i/N)	SIN(2*pi*k*i/N)		x[i] COS(2*pi*k*i/N)	x[i] SIN(2*pi*k*i/N)
0	0		1	0		0	0
1	0		1	0		0	0
2	0		1	0		0	0
3	50		1	0		50	0
4	-10		1	0		-10	0
5	-5		1	0		-5	0
6	-2.5		1	0		-2.5	0
7	-1.25		1	0		-1.25	0
8	-0.625		1	0		-0.625	0
9	-0.3		1	0		-0.3	0
10	0		1	0		0	0
11	0		1	0		0	0
12	0		1	0		0	0
13	0		1	0		0	0
14	0		1	0		0	0
15	0		1	0		0	0
					X[k]	30.325	0

RIT

Computing the DFT

- Create the COS and SINE signals at k = 1
- Then multiply by each point of the input and sum

			k=1, N=16			k=1, N=16	
i	x[i]		COS(2*pi*k*i/N)	SIN(2*pi*k*i/N)		x[i] COS(2*pi*k*i/N)	x[i] SIN(2*pi*k*i/N)
0	0		1.000	0.000		0.000	0.000
1	0		0.924	0.383		0.000	0.000
2	0		0.707	0.707		0.000	0.000
3	50		0.383	0.924		19.134	46.194
4	-10		0.000	1.000		0.000	-10.000
5	-5		-0.383	0.924		1.913	-4.619
6	-2.5		-0.707	0.707		1.768	-1.768
7	-1.25		-0.924	0.383		1.155	-0.478
8	-0.625		-1.000	0.000		0.625	0.000
9	-0.3		-0.924	-0.383		0.277	0.115
10	0		-0.707	-0.707		0.000	0.000
11	0		-0.383	-0.924		0.000	0.000
12	0		0.000	-1.000		0.000	0.000
13	0		0.383	-0.924		0.000	0.000
14	0		0.707	-0.707		0.000	0.000
15	0		0.924	-0.383		0.000	0.000
					X[k]	24.872	29.443

Computing the DFT

- Create the COS and SINE signals at k = 2
- Then multiply by each point of the input and sum

			k=2, N=16			k=2, N=16	
i	x[i]		COS(2*pi*k*i/N)	SIN(2*pi*k*i/N)		x[i] COS(2*pi*k*i/N)	x[i] SIN(2*pi*k*i/N)
0	0		1.000	0.000		0.000	0.000
1	0		0.707	0.707		0.000	0.000
2	0		0.000	1.000		0.000	0.000
3	50		-0.707	0.707		-35.355	35.355
4	-10		-1.000	0.000		10.000	0.000
5	-5		-0.707	-0.707		3.536	3.536
6	-2.5		0.000	-1.000		0.000	2.500
7	-1.25		0.707	-0.707		-0.884	0.884
8	-0.625		1.000	0.000		-0.625	0.000
9	-0.3		0.707	0.707		-0.212	-0.212
10	0		0.000	1.000		0.000	0.000
11	0		-0.707	0.707		0.000	0.000
12	0		-1.000	0.000		0.000	0.000
13	0		-0.707	-0.707		0.000	0.000
14	0		0.000	-1.000		0.000	0.000
15	0		0.707	-0.707		0.000	0.000
					X[k]	-23.541	42.063

BASIS Functions for the DFT

- Each signal can be represented by the linear combination of:
  - $N/2+1$ COSINE waves
  - $N/2+1$ SINE waves

Linear combination of N/2+1 terms

$$x[n] = \sum_{k=0}^{N/2} Re(\bar{X}[k]) \cos(2\pi ki/N) + \sum_{k=0}^{N/2} Im(\bar{X}[k]) \sin(2\pi ki/N)$$

Cosine Magnitude

Cosine Basis Function -- k

Sine Magnitude

Sine Basis Function -- k

What is $\overline{X}$?

- $X[]$ is the values that we get when we perform the DFT on the time domain signal
- $Re(X)$ is the real portion
- $Im(X)$ is the imaginary portion

- We need to scale these values when synthesizing the original signal from the SINE and COSINE signals

Scaling $Re[X]$ and $Im[X]$

$$Re(\bar{X}[k]) = \frac{Re(X[k])}{N/2}$$

$$Im(\bar{X}[k]) = -\frac{Im(X[k])}{N/2}$$

most value

Except for two special cases:

$$Re(\bar{X}[0]) = \frac{Re(X[0])}{N}$$ First frequency (DC)

$Im\ \bar{X}[0] = 0$

$$Re(\bar{X}[N/2]) = \frac{Re(X[N/2])}{N}$$ Last frequency

$Im\ \bar{X}[N/2] = 0$

Why the Scaling?

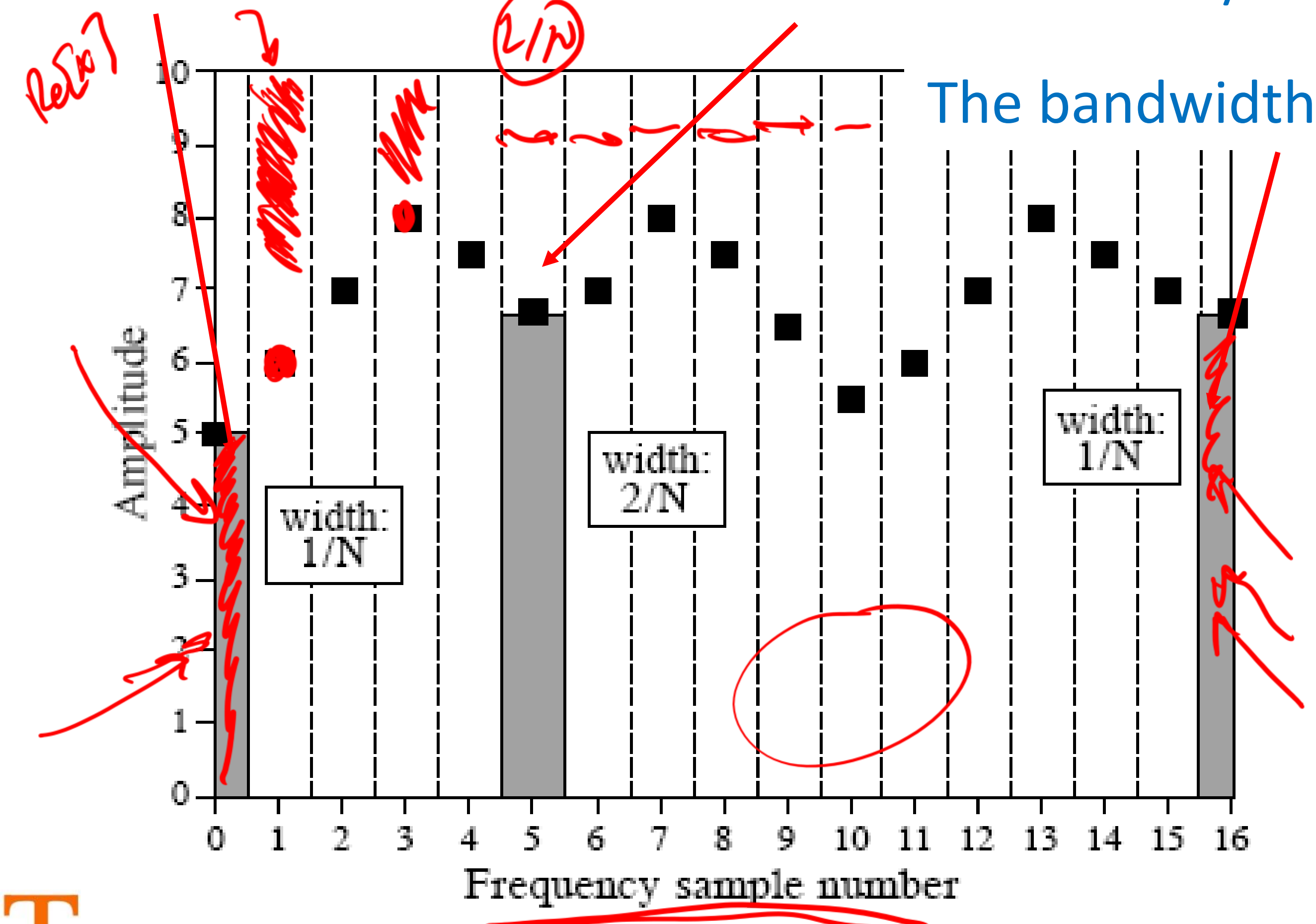
- The frequency domain coefficients are spectral densities
- Signal amplitude per unit bandwidth.
- The bandwidth is different for the end frequencies (0 and N/2)

Scaling of Coefficients

The bandwidth here is 1/N

The bandwidth here is 2/N

The bandwidth here is 1/N



Scaling $Re[X]$ and $Im[X]$

- Scale by N/2 except for $Re(X[0])$ and $Re(X[N/2])$ where the scale is N

n	Re[X]	Scale	Re[Xbar]
0	30.33	1/16	1.90
1	24.87	1/8	3.11
2	-23.54	1/8	-2.94
3	-51.36	1/8	-6.42
4	-8.13	1/8	-1.02
5	49.08	1/8	6.13
6	42.29	1/8	5.29
7	-20.09	1/8	-2.51
8	-56.58	1/16	-3.54

Special cases

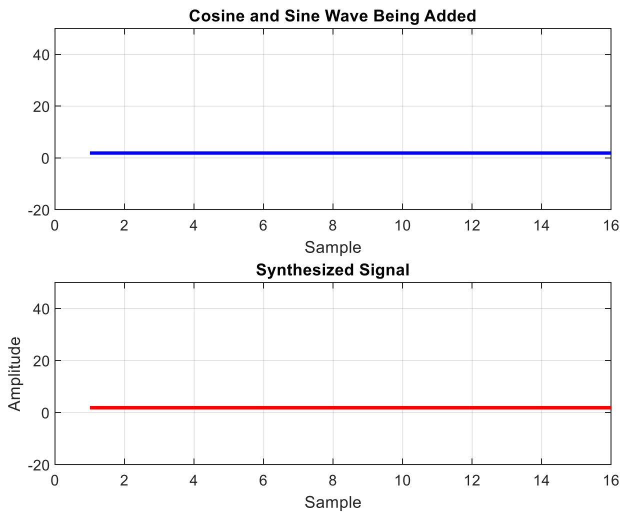
n	Im[X]	Scale	Im[Xbar]
0	0.00	~~1/8~~ 16	~~0.00~~
1	-29.44	1/8	3.68
2	-42.06	1/8	5.26
3	9.87	1/8	-1.23
4	54.05	1/8	-6.76
5	26.33	1/8	-3.29
6	-37.06	1/8	4.63
7	-52.98	1/8	6.62
8	0.00	1/8 16	0.00

Repeating Our Earlier Example
Linear Combination of COS and SINE Waves

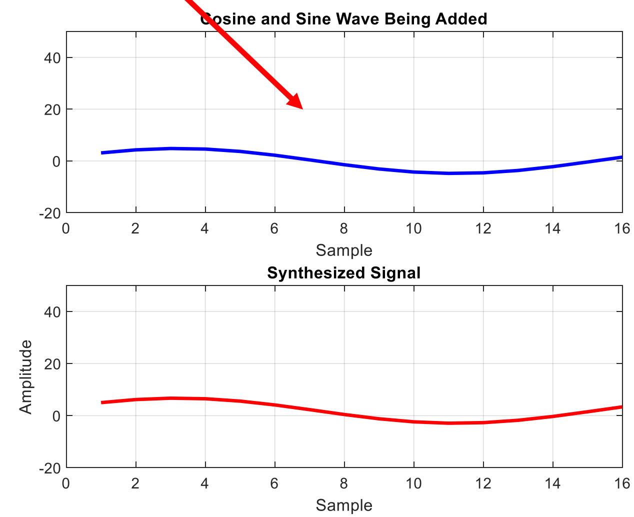
This signal is the sum of the cosine and sine at k=1 for N =16 samples

$$Re(\bar{X}[1])\cos(2\pi(1)i/N) + Im(\bar{X}[1])\ \sin(2\pi(1)/N)$$

k=0



k=1

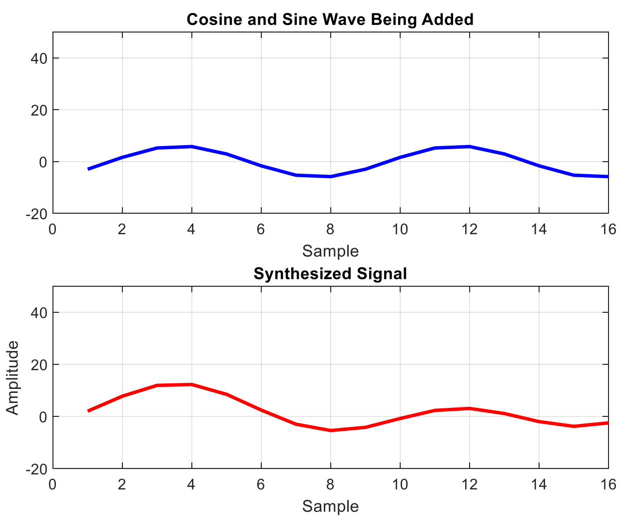


Repeating Our Earlier Example
Linear Combination of COS and SINE Waves

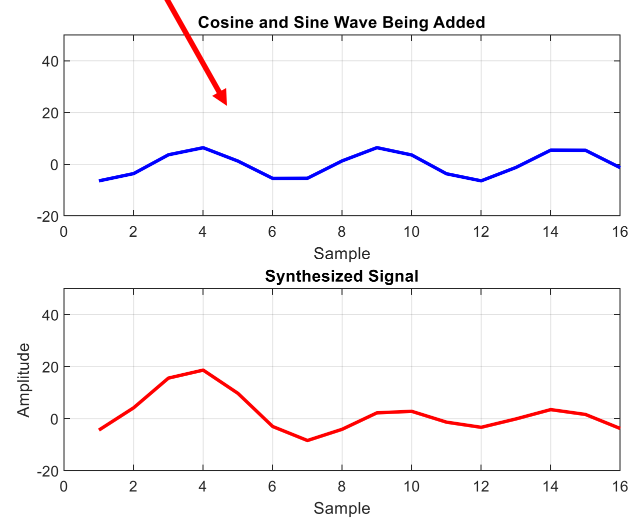
This signal is the sum of the cosine and sine at k=4 for N =16 samples

$$Re(\bar{X}[4])\cos(2\pi(4)i/N) + Im(\bar{X}[4])\,\sin(2\pi(4)/N)$$

k=3

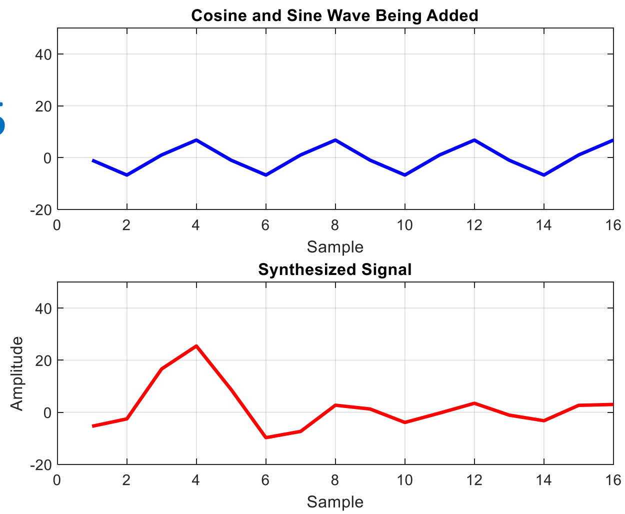


k=4

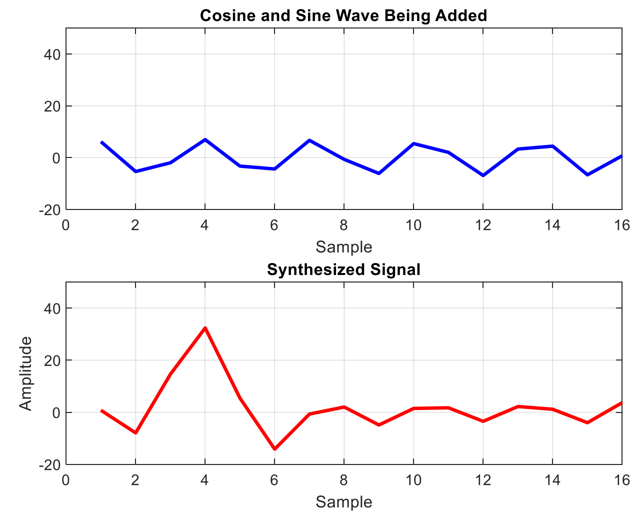


Can We Synthesize the Signal from the COS and SINE's?

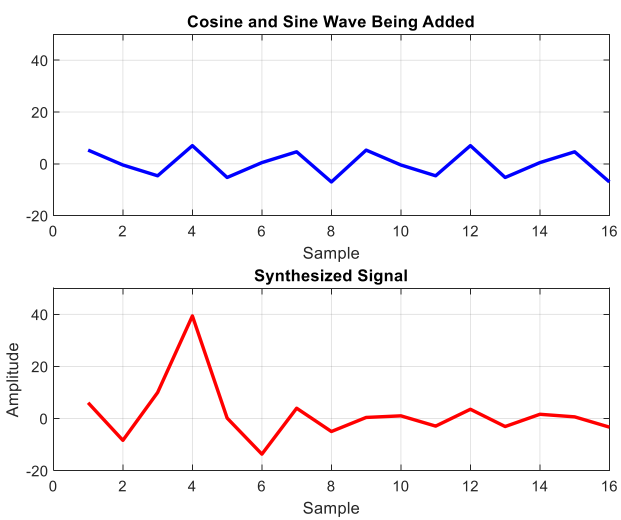
k=5



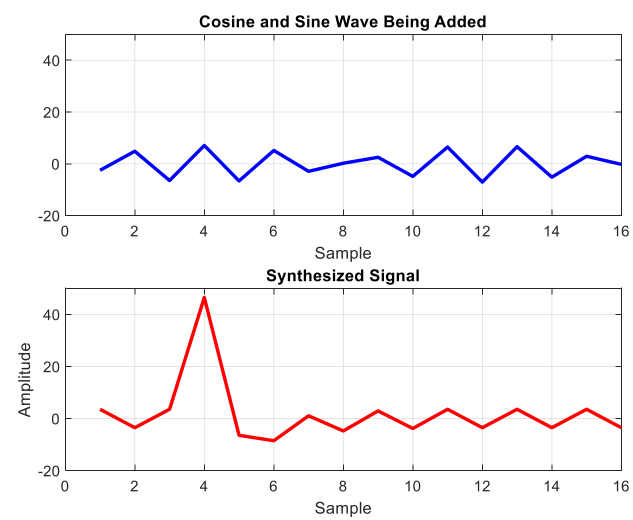
k=6



k=7



k=8

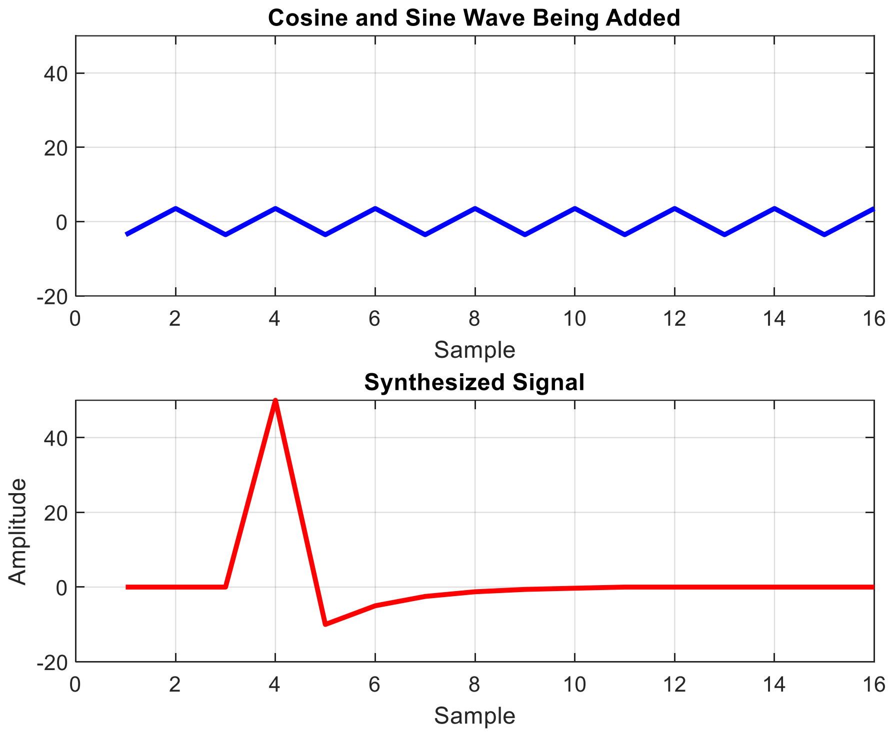


l Processing

The Original Signal Synthesized from Each COS and SINE

- The signal has been reproduced exactly from the 9 COS/SINE signal pairs

k=9



DFT ICP

- You have an input sequence [-1,2,3,1]. Compute the DFT. Use the following tables for cos and sine values. Compute $X[0], X[1], X[2]$

			k=0, N=4			k=0, N=4	
i	x[i]		COS(2*pi*k*i/N)	SIN(2*pi*k*i/N)		x[i] COS(2*pi*k*i/N)	x[i] SIN(2*pi*k*i/N)
0	-1		1	0			
1	2		1	0			
2	3		1	0			
3	1		1	0			
					X[0]		

DFT ICP

			k=1, N=4			k=1, N=4	
i	x[i]		COS(2*pi*k*i/N)	SIN(2*pi*k*i/N)		x[i] COS(2*pi*k*i/N)	x[i] SIN(2*pi*k*i/N)
0	-1		1.000	0.000			
1	2		0.000	1.000			
2	3		-1.000	0.000			
3	1		0.000	-1.000			
					X[1]		

			k=2, N=4			k=2, N=4	
i	x[i]		COS(2*pi*k*i/N)	SIN(2*pi*k*i/N)		x[i] COS(2*pi*k*i/N)	x[i] SIN(2*pi*k*i/N)
0	-1		1.000	0.000			
1	2		-1.000	0.000			
2	3		1.000	0.000			
3	1		-1.000	0.000			
					X[2]		

DFT ICP

			k=0, N=4			k=0, N=4	
i	x[i]		COS(2*pi*k*i/N)	SIN(2*pi*k*i/N)		x[i] COS(2*pi*k*i/N)	x[i] SIN(2*pi*k*i/N)
0	-1		1	0		-1	0
1	2		1	0		2	0
2	3		1	0		3	0
3	1		1	0		1	0
					X[k]	5	0

			k=1, N=4			k=1, N=4	
i	x[i]		COS(2*pi*k*i/N)	SIN(2*pi*k*i/N)		x[i] COS(2*pi*k*i/N)	x[i] SIN(2*pi*k*i/N)
0	-1		1.000	0.000		-1.000	0.000
1	2		0.000	1.000		0.000	2.000
2	3		-1.000	0.000		-3.000	0.000
3	1		0.000	-1.000		0.000	-1.000
					X[k]	-4.000	1.000

NOPRINT

DFT ICP

			k=2, N=4			k=2, N=4	
i	x[i]		COS(2*pi*k*i/N)	SIN(2*pi*k*i/N)		x[i] COS(2*pi*k*i/N)	x[i] SIN(2*pi*k*i/N)
0	-1		1.000	0.000		-1.000	0.000
1	2		-1.000	0.000		-2.000	0.000
2	3		1.000	0.000		3.000	0.000
3	1		-1.000	0.000		-1.000	0.000
					X[k]	-1.000	0.000

$$Re(X[k]) = [5, -4, -1]$$

$$Im(X[k]) = [0, 1, 0]$$

Computing the DFT Values

•

```
280 '
290 '                                   'Correlate XX[ ] with the cosine and sine waves, Eq. 8-4
300 '
310 FOR K% = 0 TO 256                   'K% loops through each sample in REX[ ] and IMX[ ]
320   FOR I% = 0 TO 511    'I% loops through each sample in XX[ ]
330     '
340     REX[K%] = REX[K%] + XX[I%] * COS(2*PI*K%*I%/N%)
350     IMX[K%] = IMX[K%]  - XX[I%] * SIN(2*PI*K%*I%/N%)
360     '
370   NEXT I%
380 NEXT K%
390 '
400 END
```

Values of the DFT

- The values of the DFT are contained in the value of X
  - Magnitudes of COSINE are $Re(\bar{X}[k])$
  - Magnitudes of SINE are $Im(\bar{X}[k])$

- We can also represent each sample in polar format

$$A cos(x) + B sin(x) = M cos(x + \theta) \quad \Longrightarrow \quad M\angle\theta$$

Polar Format

Polar Format

- For a point in the DFT

$$Mag(X[k]) = \sqrt{Re(X[k])^2 + Im(X[k])^2}$$

$$Phase(X[k]) = \arctan\left(\frac{Im(\mathrm{X[k]})}{Re(\mathrm{X[k]})}\right)$$

Polar Format

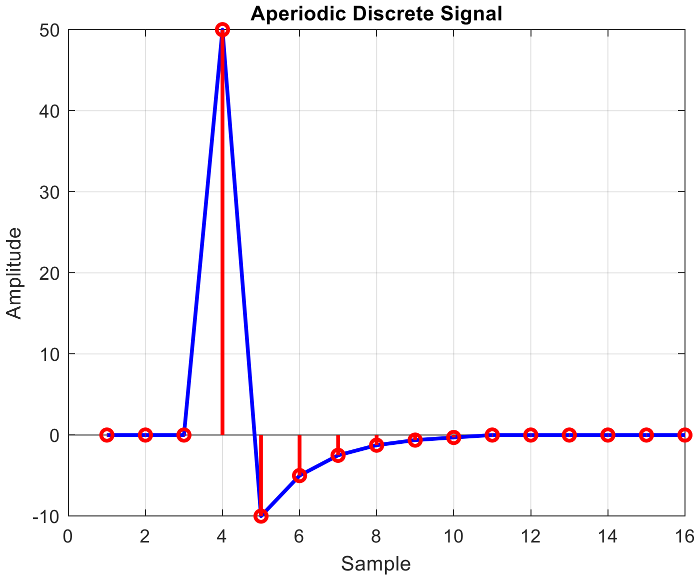
- Polar format allows us to think of the DFT in two ways

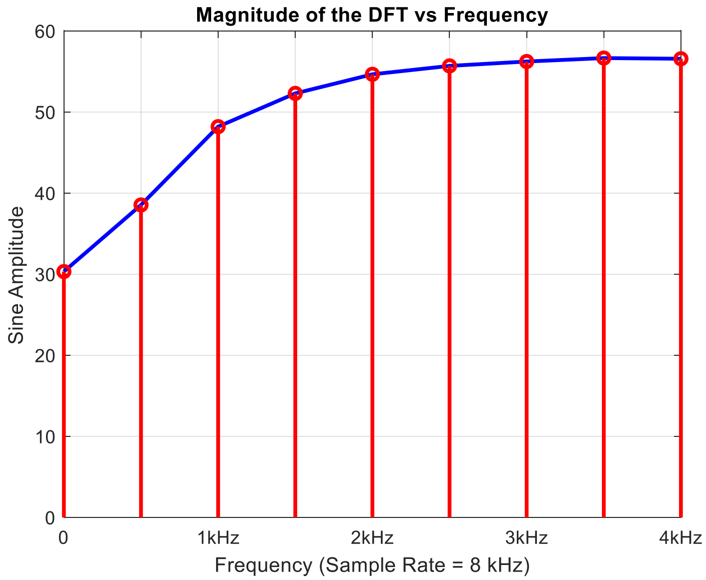
  - An N point signal decomposed into $N/2 + 1$ cosine and sine waves

  - An N point signal decomposed into $N/2 + 1$ cosine waves with a magnitude and a phase

Polar Format

- Use of the polar format allows us to look at the magnitude and phase of a signal in the frequency domain





Polar Format

- Use of the polar format allows us to look at the magnitude and phase of a signal in the frequency domain