25. L'Hôpital's rule

1. (a) Give an example of two functions $f(x)$ and $g(x)$ such that $f(0) = g(0) = 0$, and for which $\lim_{x\to 0} \frac{f(x)}{g(x)} = 5$.

   (b) Give an example of two functions $f(x)$ and $g(x)$ such that $f(0) = g(0) = 0$, and for which $\lim_{x\to 0} \frac{f(x)}{g(x)} = 0$.

   (c) Give an example of two functions $f(x)$ and $g(x)$ such that $f(0) = g(0) = 0$, and for which $\lim_{x\to 0} \frac{f(x)}{g(x)} = \infty$.

   (d) Give an example of two functions $f(x)$ and $g(x)$ such that $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = \infty$, and for which $\lim_{x\to 0} \frac{f(x)}{g(x)} = 5$.

   (e) Give an example of two functions $f(x)$ and $g(x)$ such that $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = \infty$, and for which $\lim_{x\to 0} \frac{f(x)}{g(x)} = 0$.

   (f) Give an example of two functions $f(x)$ and $g(x)$ such that $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = \infty$, and for which $\lim_{x\to 0} \frac{f(x)}{g(x)} = \infty$.

   These examples show why $0/0$ and ∞/∞ are "indeterminate form". There is no way to assign a specific, determined value to it that is consistent with every functional expression that one can write down.

2. Is $0^2/0$ indeterminate? Discuss. What about ∞/∞^2?

L'Hôpital's rule concerns the limit of quotients, like

$$\lim_{x\to 0} \frac{\sin(3x)}{x}.$$

Note that we cannot simply evaluate the argument at $x = 0$, because it is an *indeterminate form* $0/0$. Instead:

Theorem 1. *If f and g are differentiable, $f(a) = g(a) = 0$, and $g'(a) \neq 0$, then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

More generally, the following is useful if you need to apply L'Hôpital's rule more than once:

Theorem 2. *Let f and g be differentiable functions, and a a real number or $\pm\infty$. If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided the latter limit exists.

3. Use L'Hôpital's rule to find the limits:

(a) $\displaystyle\lim_{x\to 0} \frac{\sin(3x)}{x}$

(c) $\displaystyle\lim_{x\to 0} \frac{1-\cos(x)}{\sin(x^2)}$.

(b) $\displaystyle\lim_{x\to \pi} \frac{x-\pi}{\tan x}$

(d) $\displaystyle\lim_{x\to 0} \frac{x^3}{e^x - 1 - x - x^2/2}$.

4. L'Hôpital's rule also works essentially as stated for indeterminate forms ∞/∞.

Theorem 3. *Suppose f and g are differentiable. If a is a real number, or $\pm\infty$, when $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$, or $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided the latter limit exists.

5. $\displaystyle\lim_{x\to 0} \frac{1-\cos(x)}{\sin(x^2)}$. (Hint: Use Theorem 2.)

6. If f and g are two functions, then we write $f(n) \prec g(n)$ as $n \to \infty$ if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0.$$

Place the functions in increasing order with respect to the relation $\prec$ as $n \to \infty$:

$$n^2 + n, \quad n^3, \quad n + 200.$$

7. The bubble sort algorithm has average run time n^2 and the quick sort has average run time $n\log(n)$. Which of these is better on average? (First ask yourself if algorithms better if they have larger or smaller run times.)

8. The average running time of the shell sort is $O(n^{3/2})$. The average running time of the merge sort is $O(n\log n)$.

   (a) Compute $\lim\limits_{n\to\infty} \frac{n\log n}{n^{3/2}}$.

   (b) Decide whether the shell sort or merge sort has a better average running time for large values of n.

The form 0/0 is indeterminate because we have no way of knowing which 0 "wins": that is, whether the numerator or denominator tends to zero more rapidly. L'Hôpital's rule helps us to answer this question. On the other hand, there are many other expressions that can be called "indeterminate".

Here are some miscellaneous indeterminate forms to which L'Hôpital's rule can be applied. In all cases, the goal is to perform some operation that will express the indeterminate form as a quotient $\frac{\text{(something)}}{\text{(something)}}$.

To emphasize, even though there are indeterminate expressions involving other operations besides division, **L'Hôpital's rule applies *only* to quotients**. If you make up a "rule" for products/exponents/differences, then it's wrong. Many expressions can be rewritten using quotients (for example, mulitplication can be written as dividing by the reciprocal). Such tricks are sometimes needed for indeterminate forms that are not quotients.

9. "$0 \times \infty$": If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$, then we evaluate $\lim_{x\to a}(f(x)g(x))$ either by writing it as $\lim_{x\to a} \frac{f(x)}{1/g(x)}$ or $\lim_{x\to a} \frac{g(x)}{1/f(x)}$ and applying L'Hôpital's rule.

   (a) Find $\lim_{x\to 1}(x-1)\ln[(x-1)^2]$ (Hint: as a general rule, logarithms do not like to be in denominators.)

   (b) Find $\lim_{x\to\infty} x\sin(1/x)$.

10. "$\infty - \infty$": If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then we can sometimes evaluate $\lim_{x\to a}(f(x) - g(x))$ by putting $f(x)$ and $g(x)$ over a common denominator and then using L'Hôpital's rule.

(a) Calculate $\lim_{x\to 0}[(e^x - 1)^{-1} - x^{-1}]$

(b) Another trick that sometimes works for finding $\lim_{x\to a}(f(x) - g(x))$ is to multiply and divide by $f(x) + g(x)$, and then apply L'Hôpital's rule to the limit $\lim_{x\to a} \frac{(f(x)-g(x))(f(x)+g(x))}{f(x)+g(x)}$.

For example, calculate $\lim_{x\to\infty}[\sqrt{x^2 - 9x - 1} - x]$.

11. "1^∞": If $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \infty$, then we evaluate $\lim_{x\to a}(f(x)^{g(x)})$ by first taking a logarithm, which reduces it to an indeterminate form $0 \times \infty$, to which you can use the result from problem 3. For example, suppose we wish to compute the limit

$$\lim_{x\to\infty} \left(\frac{2x-3}{2x+5}\right)^{2x+1}.$$

(a) Let $f(x) = \left(\frac{2x-3}{2x+5}\right)^{2x+1}$. Expand $\ln f(x)$ using the rules for logarithms.

(b) Is $\lim_{x\to\infty} \ln f(x)$ an indeterminate form? If so, what kind?

(c) Write $\ln f(x)$ as a quotient of two functions.

(d) Use L'Hôpital's rule to compute $\lim_{x\to\infty} \ln f(x)$.

(e) Finally, find $\lim_{x\to\infty} f(x)$ by exponentiating, using the fact that $\lim_{x\to a} f(x) = e^{\lim_{x\to a} \ln f(x)}$.

12. "∞^0": If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$, then we evaluate $\lim_{x\to a}(f(x)^{g(x)})$ by first taking a logarithm, which reduces it to an indeterminate form $\infty \times 0$, to which you can use the result from problem 3. Then exponentiate the result to get the answer, using continuity of the exponential function:

$$\lim_{x\to a}(f(x)^{g(x)}) = e^{\lim_{x\to a}(g(x)\ln f(x))}$$

For example, compute $\lim_{x\to 0}\left(1+\frac{1}{x}\right)^x$.