Lecture 1: Functions

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August 28, 2018

*These slides may incorporate material from Hughes-Hallet, et al, "Calculus", Wiley

Preview

- Terms: **domain**, **range**, **linear function**, **point-slope equation**, **point-slope equation**, **exponential function**, **logarithm**
- Skills:
  - Apply the **vertical line test** to recognize the graph of a function
  - From a table of values, determine whether data is linear. Write down a linear function using the point slope equation.
  - From a table of values, determine whether data is exponential. Write down an exponential model from initial conditions (workshop).

Functions, domain, and range

Definition

A function is a rule that takes certain numbers as inputs and assigns to each a definite output number. The set of all input numbers is called the *domain* of the function and the set of all resulting output numbers is called the *range* of function.

- If the domain of a function is not specified, we usually take it to be the largest possible set of real numbers.
- Examples: $f(x) = x^2$, $g(x) = 1/x$, $\frac{1}{x^2 - x}$, $\sqrt{x+2}$
- Algebraic rules of thumb: (1) denominators cannot be zero, (2) square roots (or logarithms) cannot be evaluated at negative values.

Functions and their graphs

- The graph of a function $f(x)$ is the set of points (x, y) in the xy plane, with x in the domain of f and $y = f(x)$.
- The domain of a function is the shadow of the graph on the x-axis. The range is the shadow of the graph on the y-axis.

Vertical line test

- A curve in the *xy* plane is the graph of a function if and only if no vertical line intersects the curve more than once. (The *vertical line test*)
- Example: $x^2 + y^2 = 9$ is *not* the graph of a function

Linear functions (Example)

Model the following linear function of men's Olympic pole vault records over the period 1900–1912:

year	1900	1904	1908	1912
height (in)	130	138	146	154

- The rate of increase is the *slope*. We calculate it as

$$slope = \frac{rise}{run}.$$

- A linear function is characterized by having a constant slope. So the same "run" always will produce the same "rise".
- In this case, the slope is positive. The function is *increasing*.
- The time to run the mile since 1900 is (approximately)

$$y = g(t) = 260 - 0.39t$$

  where t is years from 1900.
- What is the slope? Is the function increasing? decreasing?

Forms of a linear function

- A linear function has the form $y = f(x) = mx + b$, where m is the slope, or rate of change of y with respect of x, and b is the y-intercept, or value of y when x is zero. (Slope-intercept form)
- A line of slope m passing through the point (x_0, y_0) is gotten by solving the equation $y - y_0 = m(x - x_0)$ for y as a function of x.
- Example: Find the linear function $y = f(x)$ such that $f(1) = 4$ and $f(3) = 1$.

Population growth

Population of Nevada:

Year	Population (millions)	Change in population (millions)
2000	2.020	
2001	2.093	0.073
2002	2.168	0.075
2003	2.246	0.078
2004	2.327	0.081
2005	2.411	0.084
2006	2.498	0.087

- Is it a linear function?
- $$\frac{\text{Population in 2001}}{\text{Population in 2000}} = \frac{2.093 \text{ million}}{2.020 \text{ million}} \approx 1.036$$
  $$\frac{\text{Population in 2002}}{\text{Population in 2001}} = \frac{2.168 \text{ million}}{2.093 \text{ million}} \approx 1.036$$
- Population grew by 36% between 2000 and 2001, and between 2001 and 2002, and similarly for the other years.

Exponential growth

- We have a constant growth factor (here 1.036) each year. The population t years after 2000 is given by the exponential function

$$P = 2.020(1.036)^t.$$

- What does the graph look like?
- To recognize that a table of t and P values comes from an exponential function $P = P_0 a^t$, look for ratios of P values that are constant for equally spaced t values.

Exponential decay

- Elimination of a drug from the body
t (hours)	0	1	2	3	4	5
Q (mg ampicillin)	250	150	90	54	32.4	19.4
- Every hour, the amount is 60% of the previous amount.
- So $Q = f(t) = 250(0.6)^t$
- Sketch $f(t)$. What does the graph look like?

Exponential functions

- An exponential function is a function of the form
$$P = P_0 a^t.$$
- P_0 is the quantity initially present (when $t = 0$)
- a is the factor by which the quantity present grows or shrinks when t is increased by 1 unit.
- If $a > 1$, then P grows exponentially (exponential growth)
- If $0 < a < 1$, then P decays exponentially (exponential decay)

Half-life and doubling

- Radioactive substances decay exponentially. Over any given unit of time, a certain percentage of the remaining radioactive matter decays, becoming non-radioactive.
- The half-life is the amount of time when exactly half of the radioactive material has decayed.
- Example: Carbon-14 has a half-life of 5730 years.

Definition

The half-life of an exponentially decaying quantity is the time required for the quantity to be reduced by half. The doubling time of an exponentially increasing quantity is the time required for the quantity to double.

Finding the doubling time

Example

A sample of 1000 bacteria is placed in a petri dish. After one hour, there are 1200 bacteria. Find the doubling time.

- The exponential model is $P = 1000(1.2)^t$
- We wish to find the value of t so that $1000(1.2)^t = 2000$.
- So $1.2^t = 2$.
- We need to bring the t down from the exponent. This requires that we use a *logarithm*.
- Taking logs on both sides of the equation, $\log(1.2^t) = \log 2$.
- By the rules for logarithms, $t \log(1.2) = \log 2$.
- Solving for t: $t = \frac{\log 2}{\log 1.2} \approx 3.80$ hours