Final exam practice workshop (solutions)

1. Suppose that $\int_0^3 f(x)\,dx = 3$, $\int_1^3 f(x)\,dx = -2$, and $\int_0^2 f(x)\,dx = 1$.

   Find $\int_1^2 f(x)\,dx$.

   Ans: $-2 + 1 - 3 = -4$

2. Using geometry, find

$$\int_{-4}^{4} \left(1 + \sqrt{16 - x^2}\right)\,dx$$

   Ans: $8 + 8\pi$

3. Evaluate the definite integral

$$\int_{-1}^{1} \left(1 + \frac{x}{\pi + x^2 + x^4}\right)\,dx$$

   (Hint: $u = x^2$.)

   Solution: We have

$$\int_{-1}^{1} \left(1 + \frac{x}{\pi + x^2 + x^4}\right)\,dx = \int_{-1}^{1} dx + \int_{-1}^{1} \frac{x}{\pi + x^2 + x^4}\,dx$$

   The first integral evaluates to 2. For the second, the change of variables $u = x^2$ transforms it to

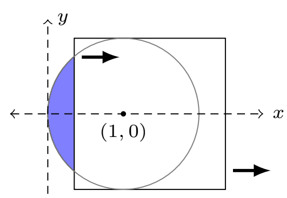
$$\int_1^1 \frac{1}{\pi + u + u^2}\,\frac{du}{2}$$

   which is zero since the limits are the same. So the answer is $2 + 0 = 2$

4. Find $\frac{d}{dx}\int_0^x \sin(t^3)\,dt$.

   Ans: $\sin(x^3)$

5. A circular pipe has a radius of 1 inch, but flow through the pipe is blocked by a valve. When the valve is open (either fully or partially, as shown below) the flow is proportional to the opened area.



   When the left-edge of the valve is at x, the flow is

$$f(x) = 2k\int_0^x \sqrt{1 - (w-1)^2}\,dw \quad \frac{\text{in}^3}{\text{sec}}$$

where k is a constant of proportionality. Suppose the valve is sliding to the right at $0.25\ \frac{\text{in}}{\text{sec}}$. What's the rate of change in the flow through the pipe when $x = 1.4\text{in}$?

Solution: We have

$$df = 2k\sqrt{1-(x-1)^2}\,dx$$

Dividing by dt gives

$$\frac{df}{dt} = 2k\sqrt{1-(x-1)^2}\frac{dx}{dt} = 2k\sqrt{1-0.4^2}\,0.25$$

6. Find the integrals

(a) $\displaystyle\int_0^4 |2x - x^2|\,dx$ Ans: $\int_0^4 (x^2 - 2x) + 2\int_0^2 (2x - x^2)\,dx =$ $4^3/3 - 4^4 + 2(2^2 - 2^3/3)$

(b) $\displaystyle\int (x^2 - 2x + 1)\,dx$ (Ans: $x^3/3 - x^2 + x + C$)

(c) $\displaystyle\int_0^1 \frac{dx}{x+1}$ (Ans: $\ln(2)$)

(d) $\displaystyle\int_{-1}^1 (x+1)\sqrt{x^2 + 2x + 1}\,dx$ (Ans: $\frac{1}{3}(x^2 + x + 1)^{3/2}\Big|_{-1}^1$

(e) $\displaystyle\int \frac{dx}{1 + 4x^2}$ (hint: $u = 2x$) (Ans: $\frac{1}{2}\arctan(2x) + C$)