Lecture 22: Three theorems about the derivative

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*These slides may incorporate material from Hughes-Hallet, et al, “Calculus”, Wiley

Fermat's theorem

Theorem

Suppose that c is a maximum or minimum of a differentiable function $f(x)$ in the interval (a, b), then $f'(c) = 0$.

Proof.

Suppose c is a maximum: $f(x) \leq f(c)$ for all x. We have

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \leq 0$$

and also

$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geq 0.$$

So $f'(c) = 0$.

If c is a minimum, the proof is the same, but with the inequalities all reversed. $\square$

- The theorem holds as stated for *local* maxima, since we can always shrink the interval if necessary.
- Application: The extreme value table.

Rolle's theorem

Theorem

Suppose that f is a continuous function on $[a, b]$ *that is differentiable on the open interval* (a, b)*. If* $f(a) = f(b)$*, then there is a* $c \in (a, b)$ *such that* $f'(c) = 0$*.*

Proof.

By the extreme value theorem,[a] the function f has an absolute maximum and an absolute minimum in $[a, b]$. Two cases:

- Case 1: Both absolute extrema are at the endpoints. Since $f(a) = f(b)$, the absolute minimum and maximum are *equal* to one another, and so f is constant: $f'(x) = 0$ at every point in that case.
- Case 2: There is an absolute extremum $x = c$ in (a, b). By Fermat's theorem, $f'(c) = 0$ at this absolute extremum □

[a] Which is very difficult to prove

Things to consider

- What happens if we relax the differentiability requirement?
- Is there a differentiable function f on an interval $[a, b]$ such that $f(a) = f(b)$, and there is *more than one point* where $f'(c) = 0$ in the interior?

The Mean Value Theorem

Theorem

If f is continuous on $a \leq x \leq b$ and differentiable on $a < x < b$, then there exists a number c with $a < c < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof.

Let $g(x) = f(x) - (f(b) - f(a))\frac{x-a}{b-a}$. Then $g(a) = f(a)$ and $g(b) = f(a)$ as well. So $g(b) = g(a)$. By Rolle's theorem, $g'(c) = 0$ for some $c \in (a, b)$. Written out, this is

$$g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} = 0$$

as required. $\square$

An example

- Let $f(x) = 1/x$ on $[1, 3]$.
- The slope of the secant line of $f(x)$ is

$$\frac{f(3) - f(1)}{3 - 1} = \frac{1/3 - 1}{2} = -\frac{1}{3}$$

- The derivative of $f(x)$ is $f'(x) = -1/x^2$.
- So, with $c = \sqrt{3}$, we have $f'(c) = -1/3$ is the slope of the secant line.
- In this case, we can find c by solving the equation

$$f'(c) = \frac{f(3) - f(1)}{3 - 1}$$

- The mean value theorem tells us that this equation *must have* a solution, even if it doesn't tell us how to find it.

Interpretations

- Geometrical interpretation: The secant line to the graph has the same slope as the tangent line at $x = c$.
- Physical interpretation: At some point of the domain of a function f, the instantaneous rate of change is equal to the average rate of change of f over its domain.
- Example: You drive on route 90 West from the Corning exit to the PA state line. The total distance is 133.6 miles, and it takes you exactly 2 hours. Your average speed was $133.6/2 = 66.8$ mph.
- At some point during the journey, your speed was *exactly* 66.8 mph.

Several corollaries

We have the *constant function theorem*:

Corollary

Suppose $f'(x) = 0$ *throughout an interval* $[a, b]$, *then* $f(x)$ *is constant.*

Proof.

If there were two points, say c and d, where $f(c) \neq f(d)$, then at some x between c and d, $f'(x) = \frac{f(c)-f(d)}{c-d} \neq 0$. This is ruled out by hypothesis, so we must have $f(c) = f(d)$ for all c, d. $\square$

Corollary

If $f'(x) = g'(x)$ *throughout* $[a, b]$, *then* $f(x) = g(x) + C$ *for some constant* C.

Proof.

Apply the constant function theorem to $f(x) - g(x)$. $\square$

The increasing function theorem

Theorem

Suppose that f is continuous on $a \leq x \leq b$ and differentiable on $a < x < b$.

- *If $f'(x) > 0$ on $a < x < b$, then f is increasing on $a \leq x \leq b$.*
- *If $f'(x) < 0$ on $a < x < b$, then f is decreasing on $a \leq x \leq b$.*

Proof.

- Suppose that $f'(x) > 0$ on $a < x < b$
- Pick x_1, x_2 with $a \leq x_1 < x_2 \leq b$
- We'll show that $f(x_1) < f(x_2)$
- We want to show $f(x_2) - f(x_1) > 0$
- The MVT implies that there is a point c between x_1 and x_2 such that $f'(c)(x_2 - x_1) = f(x_2) - f(x_1) > 0$.

□

Application: guaranteeing solutions of equations

- Let $f(x) = \sin x + 2x + 1$. Show that $f(x) = 0$ has a *unique* solution.
- We have $f(-2\pi) < 0$ and $f(0) = 1 > 0$, so there is a solution between -2π and 0 by the IVT
- Also $f'(x) = \cos x + 2 > 0$ for all real numbers x. So f is an increasing function.
- Hence the graph of $y = f(x)$ can only cross the x-axis at a single point.

Finding the solution, via Newton's method

Iterate to find the solution:

1. Start with guess x_n.
2. The next guess is $x_{n+1} = x_n - f(x_n)/f'(x_n)$.

- For $f(x) = \sin x + 2x + 1$, $f'(x) = \cos x + 2$.
- Initial guess $x_0 = 0$.
- $x_1 = 0 - f(0)/f'(0) = -1/3$
- $x_2 = -1/3 - f(-1/3)/f'(-1/3) \approx -0.335418$
- $x_3 = x_2 - f(x_2)/f'(x_2) \approx -0.33541803$
- $x_4 = x_3 - f(x_3)/f'(x_3) \approx -0.33541803238494$

So the *unique solution* to $\sin x + 2x + 1 = 0$ is
$x \approx -0.33541803238494$

The racetrack principle

Theorem

Let f and g be two continuous functions on $a \leq x \leq b$, differentiable on $a < x < b$, and that $f'(x) \leq g'(x)$ for $a < x < b$.
Hypothesis: Horse f always runs slower than horse g

1. *If $f(a) = g(a)$ then $f(x) \leq g(x)$ for $a \leq x \leq b$. If horses f and g start the race at the same time, then horse f always runs behind horse g.*
2. *If $g(b) = f(b)$ then $f(x) \geq g(x)$ for $a \leq x \leq b$. If horses f and g finish the race at the same time, then horse f would have needed a head start*

Example

Show that $e^x \geq 1 + x$ for all values of x. Let $f(x) = 1 + x$, $f(x) = e^x$. For $x \geq 0$, $f'(x) \leq g'(x)$ so use 1. For $x \leq 0$, $f'(x) \geq g'(x)$ so use 2.