Math 181A: Practice final exam, written part

1. Suppose that $\int_0^3 f(x)\,dx = 5$, $\int_1^3 f(x)\,dx = 1$, and $\int_0^2 f(x)\,dx = 1$.

Find $\int_1^2 f(x)\,dx$.

Solution. $\int_1^2 = \int_0^2 + \int_1^3 - \int_0^3 = 1 + 1 - 5 = -3$

2. Using geometry (or any other method), calculate

$$\int_{-2}^{2} \left(3 + \sqrt{4 - x^2}\right)\,dx$$

Solution. This is a Norman window whose base is a 4×3 rectangle, topped with a semicircle of radius 2. So the total area is $4\pi/2 + 12 = 2\pi + 12$

3. (a) Calculate the integral $\int_{-1}^{2} |x^2 - x|\,dx$.

Sol: We have $\int_{-1}^{2} |x^2 - x|\,dx = \int_{-1}^{2} (x^2 - x)\,dx - 2\int_0^1 (x^2 - x)\,dx$

(b) The indefinite integral $\int |x^2 - x|\,dx$ is a piecewise function. Complete the following:

$$\int |x^2 - x|\,dx = C + \begin{cases} x^3/3 - x^2/2 & \text{if } x < 0 \\ x^2/2 - x^3/3 & \text{if } 0 \le x \le 1 \\ x^3/3 - x^2/2 + 1/3 & \text{if } x > 1 \end{cases}$$

4. Evaluate the definite integral

$$\int_{-1}^{1} \frac{x\,dx}{1 + e^{x^2}}.$$

(Note: $e^{x^2} = e^{(x^2)}$, not $(e^x)^2$.)

Answer: 0

5. Calculate the integrals

(a) $\displaystyle\int \left(\sqrt{x} - \frac{2}{x} + 1\right)\,dx$

(b) $\displaystyle\int_{-1+e^{-1}}^{-1+e} \frac{dx}{x+1}$

(c) $\displaystyle\int_0^1 (x^2 + 1)\sqrt{x^3 + 3x}\,dx$

Answers: (a) $(2/3)x^{3/2} - 2\ln x + x + C$, (b) $\ln e - \ln e^{-1} = 2$, (c) $\frac{2}{9}(x^3 + 3x)^{3/2}\Big|_0^1 = \frac{16}{9}$

6. For amusement, you decide to drop a water balloon off one of the RIT buildings onto your math teacher. The velocity of the balloon is approximated by $v(t) = -32(1 - e^{-t})$ (feet/sec). If the building is 40 ft tall, write an equation whose solution is the length of time it will take the ballon to hit your poor math professor. Calculate any integrals that are involved, but do not try to solve the equation.

Solution: Let y denote the distance that the balloon travels as a function of t, so that $y(t) = \int_0^t 32(1-e^{-\tau})\,d\tau = 32(t+e^{-t}-1)$. The equation that we must solve for t is $32(t+e^{-t}-1) = 40$.

7. Compute the derivative

$$\frac{d}{dx}\int_x^{x^2} \sqrt{1-t^2}\,dt.$$

Ans: $2x\sqrt{1-x^4} - \sqrt{1-x^2}$

8. Let $F(x) = \displaystyle\int_0^x \frac{3-2t}{t^4+1}\,dt$. Find the value of x where $F(x)$ assumes its maximum value. (You do not need to evaluate the integral.)

Ans: $x = 3/2$

9. (a) Evaluate the integral

$$\int_0^1 \frac{x^4(1-x)^2}{1+x^2}\,dx$$

[Hints: By polynomial long division, $\frac{x^4(1-x)^2}{1+x^2} = x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2}$. Also, $\arctan(1) = \pi/4$ and $\arctan(0) = 0$.]

Solution: We have

$$\int_0^1 \left(x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2}\right) dx = \frac{1}{7} - \frac{4}{6} + 1 - 4/3 + 4 - 4\arctan(1) = 22/7 - \pi$$

(b) Notice that the integrand is positive, because the numerator is a perfect square and the denominator a sum of squares. It follows that this integral must be a *positive* quantity. Use your answer to (a) to prove that $\pi < 22/7$.

Solution: We have $\int_0^1 \frac{x^4(1-x)^2}{1+x^2}\,dx > 0$. By (a), this integral is $22/7 - \pi$. So $22/7 - \pi > 0$.

10. (a) Compute the integral

$$\int_0^{2\pi} \sin^2(\theta)\,d\theta$$

by pure thought. [Hint: $\cos^2\theta + \sin^2\theta = 1$]

Answer: Since $\sin^2$ and $\cos^2$ are phase shifts of each other, then have the same integral over a whole period:

$$\int_0^{2\pi} \sin^2(\theta)\,d\theta = \int_0^{2\pi} \cos^2(\theta)\,d\theta$$

Hence, taking the average of these two (equal) quantities:

$$\int_0^{2\pi} \sin^2(\theta)\,d\theta = \frac{1}{2}\int_0^{2\pi} (\sin^2(\theta) + \cos^2(\theta))\,d\theta = \frac{1}{2}\int_0^{2\pi} 1\,d\theta = \frac{1}{2}(2\pi) = \pi$$

(b) *Wirtinger's inequality* states that if $f(x)$ is a continuously differentiable function on the interval $[0,1]$ with $f(0) = f(1) = 0$, then

$$\pi^2 \int_0^1 f(x)^2\, dx \le \int_0^1 f'(x)^2\, dx.$$

Give an example of a (nonzero) function where equality (i.e., "=") holds in this inequality. Make sure you are correct by calculating both integrals. [Your answer to part (a) is relevant.]

Solution: An example is $f(x) = \sin(\pi x)$. Then we have

$$\int_0^1 \sin^2(\pi x)\, dx = \int_0^\pi \sin^2(u) du/\pi = 1/2$$

and

$$\int_0^1 f'(x)^2\, dx = \int_0^1 \pi^2 \cos^2(\pi x) dx = \pi^2/2.$$

(c) A black body in a medium held at absolute zero cools according to Newton's law of cooling, which is that the temperature $T(t)$ decays exponentially in time t. It also radiates a small amount of its heat energy in the form of black body radiation, according to the Stefan–Boltzmann law, which states that the amount of heat radiated as electromagnetic energy is

$$E(t) = \sigma A \int_0^t T(s)^4\, ds$$

where $\sigma \approx 5.7 \times 10^{-8} Js^{-1}m^{-2}K^{-4}$ is the Stefan–Boltzmann constant, and A is the surface area of the body in m^2. Assume that $A = 2m^2$. Find the rate at which electromagnetic energy radiates (i.e., the luminosity, in Js^{-1}) when the temperature is $100°K$.

Ans: We have

$$E'(t) = \sigma A T(t)^4 = (5.7 \times 10^{-8} Js^{-1}m^{-2}K^{-4})(2m^2)(100K) = 11.4 \times 10^{-6} Js^{-1}$$