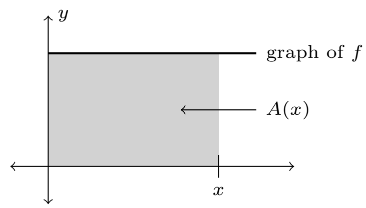
28. Antiderivatives and the fundamental theorem

1. Assume that all lengths in this exercise are measured in centimeters. Suppose $f(x) = 3$, and when $x \geq 0$ the area between the graph of f and the interval $[0, x]$ on the horizontal axis is $A(x)$.



   (a) Determine a formula for $A(x)$, including correct units.

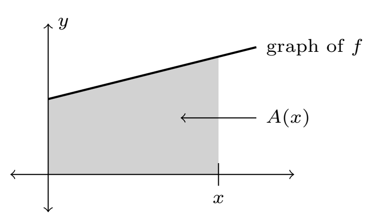
   (b) Determine $f(2)$ and $A(2)$, including correct units.

   (c) What are the units associated with $A' = \lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x} = \frac{dA}{dx}$?

   (d) Based on part (a) determine a formula for $A'(x)$ and calculate $A'(2)$

   (e) In part (d) you found an expression for A', and based on part (c) you should know what kind of geometry is described by it. Find $A'(2)$ on the graph above.

2. Assume that all lengths in this exercise are measured in inches. Suppose $f(x) = 1 + \frac{x}{4}$, and when $x \geq 0$ the area between the graph of f and the interval $[0, x]$ on the horizontal axis is $A(x)$.



   (a) Determine a formula for $A(x)$, including correct units.

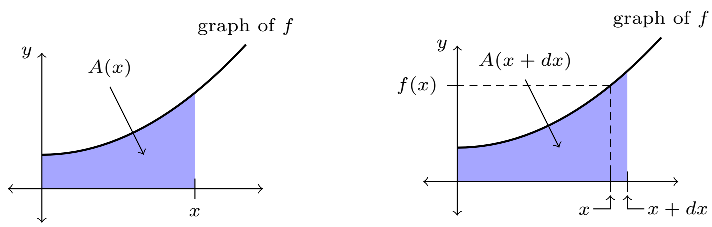
   (b) Determine $f(3)$ and $A(3)$, including correct units.

(c) What are the units associated with $A' = \lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x}$?

(d) Based on part (a) determine a formula for $A'(x)$ and calculate $A'(3)$

(e) In part (d) you found an expression for A', and based on part (c) you should know what kind of geometry is described by it. Find $A'(3)$ on the graph above.

3. Suppose f is the increasing, continuous function graphed below, and when $x \geq 0$ the area between the graph of f and the interval $[0, x]$ on the horizontal axis is $A(x)$.



Recall that $dA = A(x+dx) - A(x)$, where dx^2 (and all higher-order powers of dx) are set to zero. So $dA = A'(x)\,dx$.

(a) Shade the area in the plot that is represented by the quantity dA.

(b) You will have shaded a tall, skinny, approximately rectangular area. If we divide this area dA by its width dx, we are left with the height. What is the value of that height?

(c) Using your answer to (b), $dA/dx = ...$

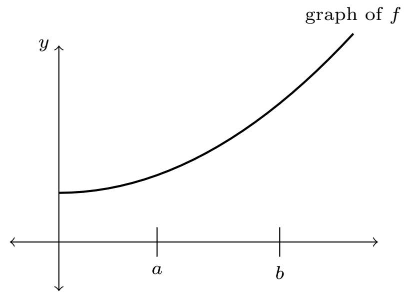
4. The previous exercises suggest that the following rule holds for the area under the graph of $y = f(x)$: $A'(x) = f(x)$. That is, the function $A(x)$ is an *antiderivative* of $f(x)$. Sometimes the best way to find an antiderivative is by guessing a function.

   (a) Find an antiderivative of the function $f(x) = 1 + x/4$.

   (b) Find an antiderivative of the function $f(x) = x^2$.

   (c) Find an antiderivative of the function xe^x. The easiest way to do this is called the *method of undetermined coefficients*: assume that the answer has the form $F(x) = (Ax + B)e^x$, and impose $F'(x) = f(x)$ to solve for the unknown constants A and B.

5. The plot below shows values a and b on the x axis.



(a) Illustrate the areas $A(a)$ and $A(b)$.

(b) In terms of $A(a)$ and $A(b)$, write down a formula for the area under the graph of $y = f(x)$ that above the segment $a \leq x \leq b$ of the x-axis.

The area that you found is called the *definite integral*, and we denote it by

$$\int_a^b f(x)\,dx.$$

6. The combination of the last two exercises gives us a way to find the area under a graph. Suppose that $A(x)$ is an antiderivative of the function $f(x)$. Then the *fundamental theorem of calculus* says that the area under the graph of $y = f(x)$ over the segment $a \leq x \leq b$ of the x-axis is given by:

$$\int_a^b f(x)\,dx = A(b) - A(a).$$

We can use this to find the area under a graph. For example, consider the region under the graph of $y = f(x) = 1 + \frac{x}{4}$, over the interval $1 \leq x \leq 3$.

(a) Sketch this region in the xy-plane. The region is a trapezoid. What is its area?

(b) Alternatively, we find an antiderivative of $f(x) = 1 + \frac{x}{4}$ to be $A(x) = x + \frac{x^2}{8}$. Then we can calculate the area using the fundamental theorem of calculus by

$$\int_1^3 f(x)\,dx = A(3) - A(1) = (3 + 9/8) - (1 + 1/8) = 3$$

which (hopefully) agrees with your answer to (a).

7. Use the fundamental theorem of calculus to find the area under the graph of $y = x^2$ over the interval $2 \leq x \leq 3$.