Lecture 19: The chain rule

Jonathan Holland

Rochester Institute of Technology*

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* These slides may incorporate material from Hughes-Hallet, et al, “Calculus”, Wiley

Differentials of composite functions, first method

- The rules for computing d(function) propagate down the parse tree using a list of tree-rewriting rules.
- Example: Compute $d(\cos(x^2+1))$. Parse tree:

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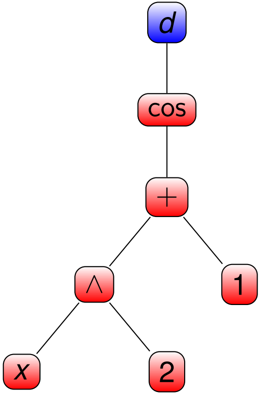

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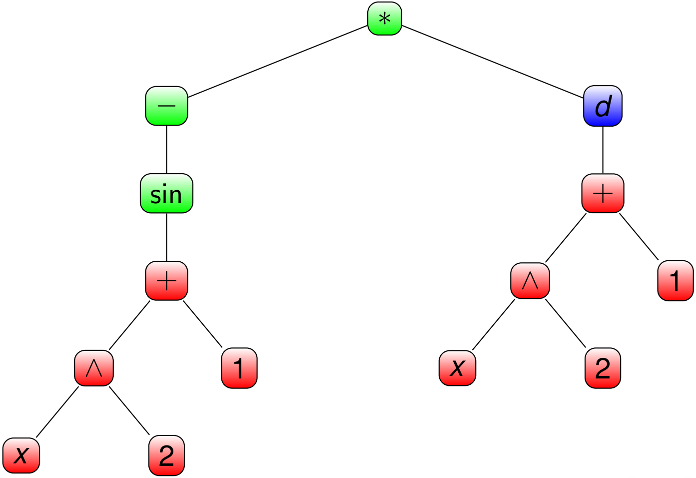

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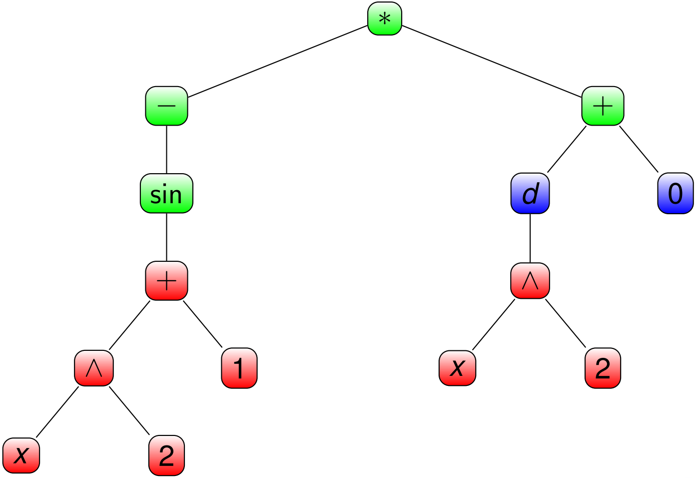

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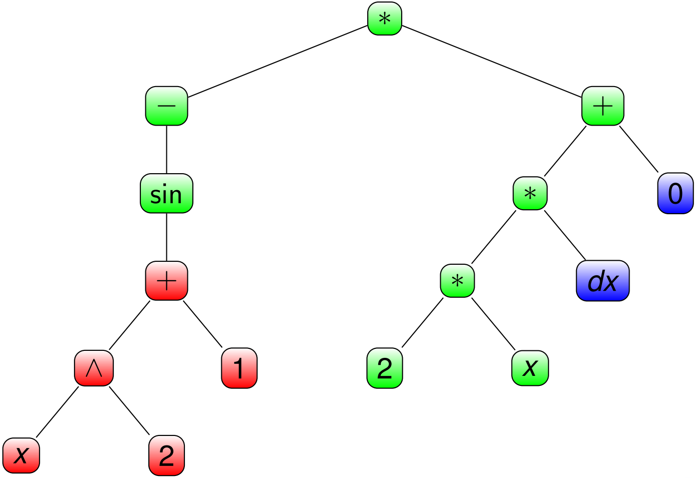

Differentials of composite functions, first method

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d(\cos(x^2+1)) &= d(\cos u) \\
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A second example

Problem

Compute the differential of $y = (x^2 + 1)^{10}$.

- We parse the expression from the outside in.
- Note that the power is the *outside function.*
- Use the power rule

$$d(u^{10}) = 10u^9\,du$$

with $u = x^2 + 1$ (the *inside function*)

- So

$$\begin{aligned} d(x^2+1)^{10} &= 10(x^2+1)^9 d(x^2+1) \\ &= 10(x^2+1)^9\,(2x\,dx) \\ &= 10(x^2+1)^9 2x\,dx \end{aligned}$$

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Another look: Dependency diagrams and the chain rule

- A *dependency diagram* is a simplified parse tree that allows us to express a complicated function as a composite of several simpler functions.

$$\begin{array}{c} y = u^{10} \\ | \\ u = x^2 + 1 \\ | \\ x \end{array}$$

- Read from the bottom up: beginning from x, how can I build up the function y in several simpler steps?
- The *chain rule* says how to calculate the *derivative* dy/dx:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

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The derivative of a composite function

- Suppose that $z = g(x)$ and $y = f(z)$, so $y = f(g(x))$.
- We calculate
$$dy = f'(z)\,dz = f'(g(x))g'(x)\,dx$$
- Or, stated another way, $\dfrac{dy}{dx} = \dfrac{dy}{dz}\dfrac{dz}{dx}$.

Theorem

If f and g are differentiable functions, then
$$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x).$$
In words: The derivative of a composite function is the product of the derivatives of the outside and inside functions. The derivative of the outside function must be evaluated at the value of the inside function.

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An example

- Example: Climbing a mountain, the air temperature H depends on elevation y. That is, $H = f(y)$.
- The rate of change of the air temperature is affected by how fact the temperature changes with altitude (about $-3.3°F$ for every 1000 feet), and by how fast we are climbing (say $500 ft/h$).
- So the rate of change of air temperature with respect to time is
$$\frac{-3.3°F}{1000 ft} \times \frac{500 ft}{hr} = -1.15°F/hr$$
- Notice that the units of the final answer ($°F/hr$) already tell us what we need to do: multiply something with units of $°F/ft$ by something with units of $°ft/hr$.

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Intuition behind the chain rule

- Since temperature is a function of height $H = f(y)$ and height is a function of time $y = g(t)$, we can think of temperature as a composite function of time $H = f(g(t))$, with f as the outside function and g the inside function.
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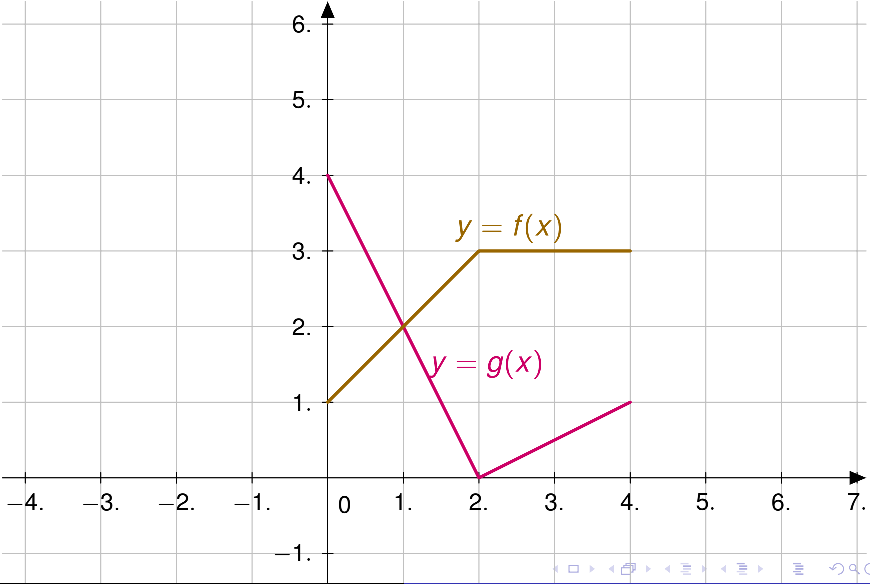
Graphical example

Find $\frac{d}{dx}(f(g(x)))|_{x=3}$



Algebraic examples

- $(x^2 + 1)^{100}$
- $\sqrt{x^2 + 5x - 2}$
- $\frac{1}{x^2+x^4}$
- e^{3x}
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