Lecture 16: The product and quotient rules

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*These slides may incorporate material from Hughes-Hallet, et al, “Calculus”, Wiley

Review

Rules

Let u, v be any expressions, and c a constant. Then

- Power rule: $d(u^p) = pu^{p-1}\,du$
- Sum rule: $d(u+v) = du + dv$
- Constant multiple rule: $d(cu) = c\,du$

Examples:

- $d(x^2) = ...$
- $d(\sqrt{t}) = ...$
- $d(u^\pi) = ...$
- $d(y^2) = ...$
- $d(x^2 + y^2) = ...$

Bird gets food...

Using the differential to find a tangent line

Problem

Find the equation of the tangent line to $x^2 + y^2 = 25$ at $(-3, 4)$.

Solution

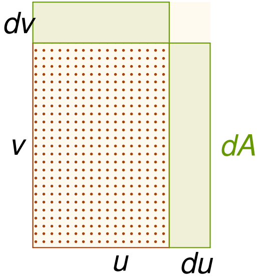
We take differentials of both sides of the equation $x^2 + y^2 = 25$:

$$\begin{aligned} d(x^2 + y^2) &= d(25) \\ d(x^2) + d(y^2) &= 0 && \textit{sum rule and constant rule} \\ 2x\,dx + 2y\,dy &= 0 && \textit{power rule} \end{aligned}$$

Now plug in $x = -3$,$y = 4$,$dx = x - (-3)$,$dy = y - 4$ to get the equation of the tangent line:

$$2(-3)(x - (-3)) + 8(y - 4) = 0 \implies -3x + 4y = 25.$$

Area of a rectangle



- Suppose we have a rectangle of sides u and v. The area is $A = uv$.
- Now, suppose we independently increase u by a small amount du and v by a small amount dv.
- Then $(u + du)(v + dv) = uv + u\,dv + v\,du + du\,dv$
- If du,dv are both sufficiently small, then for practical purposes we may set $du\,dv = 0$.
- Thus

$$dA = (u + du)(v + dv) - uv = u\,dv + v\,du.$$

The product rule

Product rule

If u and v are expressions, then $d(uv) = u\,dv + v\,du$.

Example:

- $d(x^2(x^3 - 2x)) = d(x^2)\,(x^3 - 2x) + x^2\,d(x^3 - 2x) = 2x(x^3 - 2x)\,dx + x^2(3x^2 - 2)\,dx$
- So, if $f(x) = x^2(x^3 - 2x)$, then $f'(x) = df/dx = 2x(x^3 - 2x) + x^2(3x^2 - 2)$

Example: Ohm's law

- Ohm's law in physics states that the voltage drop V across a resistor R satisfies $V = IR$, where I is the current.
- So $dV = I\,dR + R\,dI$.
- If the resistor is subjected to a constant voltage, then $dV = 0$.
- In that case, $I\,dR + R\,dI = 0$, or $\frac{dR}{R} + \frac{dI}{I} = 0$.
- As a consequence, for example, if we increase the resistance by 1% (so $dR/R = 0.01$), then the current will fall by 1% ($dI/I = -0.01$).

Rules for differentials $\leftrightarrow$ rules for derivatives

- If $u = f(x)$ is a function of x, then $du = f'(x)dx$.
- So any rule for d has a corresponding rule for the derivative: we just need to divide by dx.
- If $u = f(x)$ and $v = g(x)$, then by the product rule, $d(uv) = du\, v + u\, dv$.
- So

$$\frac{d(uv)}{dx} = \frac{du}{dx}\, v + u\, \frac{dv}{dx} = f'(x)g(x) + f(x)g'(x)$$

Product rule for three functions

- If u, v, w are three expressions, then

$$d(uvw) = du\,vw + u\,dv\,w + uv\,dw.$$

- Example:

$$\begin{aligned}\frac{d}{dx}[x^2(x+1)(x+2)] &= \frac{d}{dx}(x^2)\,(x+1)(x+2)+\\ &\quad + x^2\frac{d}{dx}(x+1)\,(x+2)+\\ &\quad + x^2(x+1)\frac{d}{dx}(x+2)\\ &= 2x(x+1)(x+2)\,\frac{dx}{dx} + x^2(x+2)\,\frac{dx}{dx}+\\ &\quad + x^2(x+1)\,\frac{dx}{dx}\\ &= 2x(x+1)(x+2) + x^2(x+2) + x^2(x+1)\end{aligned}$$

Quotient rule

Quotient rule

If u, v are expressions, then

$$d\left(\frac{u}{v}\right) = \frac{du\,v - u\,dv}{v^2}$$

Proof.

If $y = u/v$, then $u = yv$. By the product rule,

$$du = d(yv) \implies du = dy\,v + y\,dv.$$

Solving for dy gives

$$\begin{aligned} dy &= \frac{du - y\,dv}{v} = \frac{du - \frac{u}{v}dv}{v} \\ &= \frac{du\,v - u\,dv}{v^2}. \end{aligned}$$

Quotient rule for derivatives

Example

Let $y = \frac{1}{x^2+1}$. Then

$$dy = \frac{d(1)\,(x^2+1) - (1)d(x^2+1)}{(x^2+1)^2} = \frac{-2x\,dx}{(x^2+1)^2}.$$

So $dy/dx = \frac{-2x}{(x^2+1)^2}$