29. Finding integrals by the fundamental theorem

1. Suppose $q(t)$ is the number of calls received per hour, t hours after a company's customer service center opens at 8:00 am each day. Use the definite integral to write an equation that says, "The customer service center received 205 calls between 9 am and noon."

   Answer: $\int_1^4 q(t)\,dt = 205$

2. It's often helpful to stratify a population based on age—newborns, toddlers, ..., senior citizens, centenarians—because diseases (among other things) affect them differently. Suppose the population of a city is $\int_0^{150} f(a)\ da$ people, where a is measured in years (and $f(a)$ is zero when a is large). What are the units associated with $f(a)$ and what does it mean to us?

   Answer: $f(a)$ is people per year, and it represents the number of people of age a.

3. Suppose the net electric charge on a rod is $\int_0^1 f(x)\ dx$ coulombs, where x is measured in meters. What are the units associated with $f(x)$ and what does it mean to us?

   The units are coulombs per meter. This is a (one-dimensional) charge density.

4. Once again, the horses Sweetie and Thunder are racing. Thunder is faster than Sweetie. (Sweetie is more lovable, but that is not relevant for the problem.) Let $S(t)$ be Sweetie's speed, and $T(t)$ Thunder's speed, both measured in meters per second. We know from the properties of integrals that

$$S(t) \leq T(t) \text{ on } [0,60] \quad \Rightarrow \quad \int_0^{60} S(t)\ dt \leq \int_0^{60} T(t)\ dt.$$

   Explain what this means in the context of the running horses.

   If Sweetie runs more slowly than Thunder, then she also runs a shorter distance in the same amount of time.

5. Solve the puzzle: Suppose $3x + 7 = \int_a^x f(t)\ dt$. What is the value of a?

   $a = -7/3$

6. Find a value of $T > 0$ so that the signed area between the graph of $f(x) = 3x^2 - 4$ and the x-axis over $[0, T]$ is zero.

   $\int_0^T (3x^2 - 4)\,dx = x^3 - 4x\Big|_0^T = T^3 - 4T$

   So we want $T^3 - 4T = 0$. Solving for T gives $T = 0, 2$. Of these, only $T = 2$ is positive.

7. Integration is hard. Differentiation is easy. Often integration can be so hard that the best way to calculate an integral is just to guess the answer, and then verify that it is correct. In this example, you will calculate

$$\int_0^{\ln(2)} xe^x\,dx.$$

   (a) First we need to find an antiderivative of xe^x, that is a function $F(x)$ such that $F'(x) = xe^x$. Based on algebraic intuition, we guess that the form of the integral should be a linear function times the exponential function. So, we write down $F(x) = (Ax + B)e^x$. Calculate $F'(x)$, being careful to use the product rule correctly.

   (b) Your answer to (a) can be factored as $F'(x) = (\text{a linear function})e^x$. Set this equal to xe^x and solve for the unknown constants A and B.

   (c) Finally, calculate the definite integral $\int_0^{\ln(2)} xe^x\,dx$ using the fundamental theorem of calculus.

   (a) $F(x) = (Ax + B)e^x \implies F'(x) = (Ax + A + B)e^x$. Setting this equal to xe^x we have $A = 1$ and $B = -1$. So $F(x) = (x - 1)e^x$ is an antiderivative. (b) $\int_0^{\ln(2)} xe^x\,dx = F(\ln(2)) - F(0) = (\ln 2 - 1)e^{\ln 2} - (0 - 1)e^0 = 2(\ln 2 - 1) + 1 = 2\ln 2 - 1$

8. An *exponentially damped* oscillator can be put into the form $f(t) = e^{-t}\cos t$. Our physical intuition tells us that $f(t)$ will have an antiderivative $F(t)$ that is also exponentially damped, but possibly with a phase shift. So we guess an antiderivative of the form $F(t) = (A\cos t + B\sin t)e^{-t}$ for some undetermined coefficients A and B. Determine these constants by solving $F'(t) = f(t)$.

   With $F(t) = (A\cos t + B\sin t)e^{-t}$, we have $F'(t) = (-A + B)\cos te^{-t} + (-A - B)\sin te^{-t}$. Setting this equal to $e^{-t}\cos t$, we have $(-A + B) = 1$ and $(-A - B) = 0$. So $A = -B = -1/2$. Thus $F(t) = (-\frac{1}{2}\cos t + \frac{1}{2}\sin t)e^{-t}$

9. Calculate the following integrals using the Fundamental theorem. (You may use the "rules" for integration that you learned today, or the guess-and-check method.)

   (a) $\displaystyle\int_0^1 e^x \, dx$

   (b) $\displaystyle\int_1^2 (x+1)^2 \, dx$

   (c) $\displaystyle\int_1^3 x^{-2} \, dx$

   (d) $\displaystyle\int_1^e \frac{1}{x} \, dx$ [Hint: what function has derivative equal to $1/x$?]

   (a) $e^1 - e^0 = e - 1$, (b) $\frac{1}{3}(x+1)^3\Big|_1^2 = \frac{1}{3}3^3 - \frac{1}{3}2^3$, (c) $-x^{-1}\Big|_1^2 = -2^{-1} - (-1^{-1}) = 1/2$

10. Find a value of $T > 0$ so that the net signed area between the graph of $f(x) = 3x^2 - 4$ and the x-axis over $[0, T]$ is zero.
    $\int_0^T (3x^2 - 4)\, dx = x^3 - 4x\Big|_0^T = T^3 - 4T$
    So we want $T^3 - 4T = 0$. Solving for T gives $T = 0, 2$. Of these, only $T = 2$ is positive.

11. Suppose a computer chip cools at a rate of $T'(t) = -0.5e^{-0.1t}$ °F/sec after the machine is turned off.

    (a) Write down an integral that gives the answer to the following question: What is the net change in the chip's temperature during the first minute after the machine is turned off?

    (b) Calculate that integral using the Fundamental theorem of calculus. [Hint: Guess an antiderivative of the form $Ae^{-0.1t}$.]

    (c) If the chip's operating temperature is 85°F, how long does it take to cool down to 81°F?

    Answer: (a) $\int_0^{60} -0.5e^{-0.1t}\, dt$ (b) $5e^{-0.1t}\Big|_0^{60} = -5(1 - e^{-6})$°F

    (c) $t = 10\ln(5) \approx 16.094$ seconds