Lecture 10: Growth and dominance

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*These slides may incorporate material from Hughes-Hallet, et al, “Calculus”, Wiley

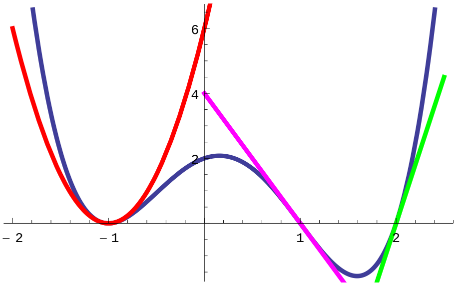
Preview

- Continuity, definition in terms of limits. $f(x)$ is continuous at $x = c$ if c is in the domain of f, and $\lim_{x\to c} f(x) = f(c)$.
- Continuity, as a statement about approximation
- A function $g(x)$ is **ultimately much bigger than** $f(x)$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$. We write "$f \prec g$ as $x \to \infty$"
- A suppose two functions $f(x), g(x)$ tend to zero as $x \to c$. We say that $f(x) \prec g(x)$ as $x \to c$ if $\lim_{x\to c} \frac{f(x)}{g(x)} = 0$.
- We write $f(x) \approx g(x)$ as $x \to c$ if $f(x) - g(x) \prec g(x) - g(c)$

Continuity

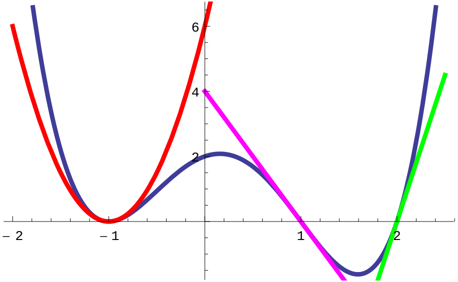
- Suppose x represents a measurement, whose true value is $x = c$, but that has some small error Δx.
- Let $y = f(x)$ be a computation we must perform on the measurement.
- The function is continuous at $x = c$ provided that the error in the computation, $\Delta y = f(x) - f(c)$ is comparable to the error in the measurement, $\Delta x = x - c$.
- That is, for any target accuracy ϵ in the computation, $|\Delta y| < \epsilon$, we can make the measurement accurate enough to achieve this target.
- In other words, for all $\epsilon > 0$, there exists a $\delta > 0$ such that $|\Delta y| < \epsilon$ provided $|\Delta x| < \delta$.

Review: localization of a polynomial at a zero



- Example: $f(x) = (x-1)(x+1)^2(x-2)$.
- Near $x = 1$, $f(x) \approx (x-1)(1+1)^2(1-2) = -4(x-1)$ (purple)
- Near $x = -1$, $f(x) \approx (-1-1)(x+1)^2(-1-2) = 6(x+1)^2$ (red)
- Near $x = 2$, $f(x) \approx (2-1)(2+1)^2(x-2) = 9(x-2)$ (red)

Being more precise about $\approx$



- $f(x) = (x-1)(x+1)^2(x-2)$.
- When we write $f(x) \approx 6(x+1)^2$ near $x = -1$, we mean that $f(x) = 6(x+1)^2 + e(x)$ where $e(x)$ is an error that tends to zero faster than $(x+1)^2$.
- That is, $\lim_{x \to -1} \frac{e(x)}{(x+1)^2} = 0$.
- We can check that this works, because we have an explicit formula for the error $e(x)$:
  $e(x) = (x-1)(x+1)^2(x-2) - 6(x+1)^2 =$
  $(x+1)^2((x-1)(x-2)-6) = (x+1)^3(x-4)$

Order of smallness

- For values of x near 0, which is smaller x, x^2, x^3?

Definition

We say that $x^2 \prec x$ as $x \to 0$ if

$$\lim_{x \to 0} \frac{x^2}{x} = 0.$$

- This says that x^2 *tends to zero faster* than x.
- Similarly $x^3 \prec x^2$ as $x \to 0$.

Theorem

If $n < m$, then $x^m \prec x^n$ as $x \to 0$.

Definition of $\approx$

Definition

Let $f(x)$ and $g(x)$ be continuous functions. We will say that $f(x) \approx g(x)$ for x near a if

$$f(x) = g(x) + e(x)$$

where $e(x)$ tends to zero faster than $g(x) - g(a)$. Equivalently, $f(x) \approx g(x)$ near $x = a$ if $f(a) = g(a)$ and

$$\lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} = 1.$$

Example

- $\sin x \approx x$ for x near 0.
- $e^x \approx 1 + x + x^2/2$ for x near 0: $e^0 = 1 = 1 + 0 + 0^2/2$, so the approximation is valid if $\lim_{x \to 0} \frac{e^x - 1}{x + x^2/2} = 1$ (L'Hôpital)

Comparing functions

- We say that a function $f(x)$ **tends to zero faster than** $g(x)$ as $x \to a$ if $\lim_{x\to a} \frac{f(x)}{g(x)} = 0$. We write "$f(x) \prec g(x)$ as $x \to a$"
- Localization: We will write "$f(x) \approx g(x)$ for x near a" if $f(a) = g(a)$ and $f(x)$ tends to $f(a)$ at the same rate that $g(x)$ tends to $g(a)$:

$$\lim_{x\to a} \frac{f(x) - f(a)}{g(x) - g(a)} = 1.$$

- The **linearization** of a function $f(x)$ is the linear function as $L(x) = f(a) + m(x - a)$ that satisfies $f(x) \approx L(x)$ for x near a.

The approximation problem

- A central application of calculus is *finding useful approximations* of functions, and also *estimating the error* in those approximations.
- For example, suppose we measure the side of a square to be $x = 1in$, so that the square has area $x^2 = 1in^2$.
- Suppose that there is a small error $dx = 0.1$ in our measurement, so that the true value of the side is actually $x + dx = 1.1in$.
- The true value of the area is then $(x + dx)^2 = 1.21in^2$. So there is an error in the area of $0.21in^2$.

The approximation problem continued

- More generally, if we compute a side length of $x = c$ and we can estimate that there is a possible error in the length dx, then we can estimate what the error in the area is by

$$\begin{aligned}(c + dx)^2 &= c^2 + 2c\,dx + dx^2 \\ &= c^2 + (2c + dx)\,dx\end{aligned}$$

- The error term is *comparable* to the error dx in the original measurement.

Limits as approximations

- Suppose that f is a function, and we are interested in values of the function near $x = c$.
- Suppose that for small values of dx, we have the approximation $f(c + dx) \approx L$. If the error in this approximation is comparable to dx, then we say that the limit of $f(x)$ is L as x approaches c.
- What does it mean for these errors to be comparable?
- The error in the approximation $f(c + dx) \approx L$ is the difference $|f(x + dx) - L|$.
- We shall say that this error is *comparable* to dx if it can be made as small as desired by making dx sufficiently small.

Dominance

- $\lim_{x\to\infty} x = \ldots\infty$
- $\lim_{x\to\infty} x^2 = \ldots\infty$
- Which of the functions $f(x) = x$ and $g(x) = x^2$ is *ultimately much bigger*?
- Note that it makes no sense to compare "∞" with "∞^2", because these are not numbers!
- The key is that the limit

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$$

- This tells us the growth in $g(x)$ is ultimately much larger than that of $f(x)$. (Numerical example: Consider $\frac{f(10)}{g(10)} = \frac{10}{10^2} = \frac{1}{10}, \frac{f(100)}{g(100)} = \frac{100}{100^2} = \frac{1}{100}$ and so forth.)

Limit laws at infinity...

Assuming all the limits on the right hand side exist (and are finite):

1. If b is a constant, then $\lim_{x\to\infty}(bf(x)) = b\lim_{x\to\infty} f(x)$ (contant multiple law)
2. $\lim_{x\to\infty}(f(x)g(x)) = \lim_{x\to\infty} f(x) \lim_{x\to\infty} g(x)$ (product law)
3. $\lim_{x\to\infty}(f(x)+g(x)) = \lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x)$ (sum law)
4. $\lim_{x\to\infty}(f(x)/g(x)) = \lim_{x\to\infty} f(x)/\lim_{x\to\infty} g(x)$, provided $\lim_{x\to\infty} g(x) \neq 0$ (quotient law)
5. $\lim_{x\to\infty}[f(x)]^n = [\lim_{x\to\infty} f(x)]^n$ if n is a positive integer (power law)
6. $\lim_{x\to\infty} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to\infty} f(x)}$ provided $f(x) > 0$ for n even (root law)
7. $\lim_{x\to\infty} k = k$ for any constant k.
8. $\lim_{x\to\infty} \frac{1}{x^n} = 0$ for $n > 0$

Defining "Ultimately much bigger than"

Definition

A function $g(x)$ is *ultimately much bigger than* $f(x)$, written $f \prec g$ at $x \to \infty$, if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$

Example

- Algorithm run-times (or space): Often write "$f(n) = o(g(n))$" for $f \prec g$. For example, the *quicksort* has average performance $o(n)$ in the size n of an array.
- $\preccurlyeq$, $O(n)$