Lecture 15: The power rule

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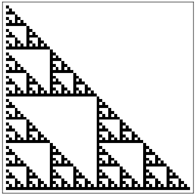
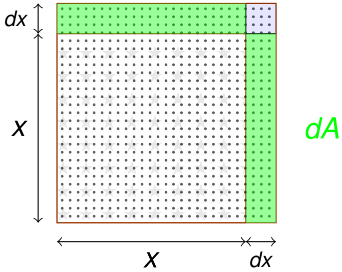


Figure: The parity (evenness or oddness) of the numbers appearing in Pascal's triangle

Preview

- The **power rule** states that, if $y = x^p$ is a power function, then
$$dy = px^{p-1}\,dx.$$
- Interpret the differentials $d(x^2)$ and $d(x^3)$ geometrically and algebraically.
- Distinguish between the differential, dy, and the derivative dy/dx:
  - dx and dy are variables, representing a small change in the x variable, and a small compensating change in the y variable.
  - the ratio dy/dx represents the rate at which y changes with respect to changes in the x variable

Differential of an area

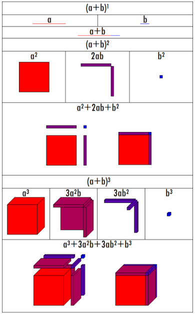


- The area of a square of side x is $A = x^2$.
- If we increase x by a little bit dx, how much does the area increase? We call this dA.
- We have $dA = (x + dx)^2 - x^2 = x^2 + 2x\,dx + dx^2 - x^2 = 2x\,dx + dx^2 = 2x\,dx$

Differential of a volume

Suppose the side length of a cube is x.

- Write the formula for the volume of the cube, V.
- Fill in the blanks $(x + dx)^3 = x^3 +$ ______________ $3x^2\,dx$
- So $dV = 3x^2\,dx$. (Animation)
- Note $\frac{dV}{dx}$ is half the surface area of the cube: when we add dx to each of the sides, half the faces of the cube get fattened out by $x \times x \times dx$ slabs.



The differential versus the derivative

- Let's look again at the calculation of $d(x^2)$:
  $d(x^2) = (x + dx)^2 - x^2 = x^2 + 2x\,dx + dx^2 - x^2 =$
  $2x\,dx + dx^2 = 2x\,dx$
- Now, let's calculate the derivative of the function $f(x) = x^2$:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2}{h} \\
&= \lim_{h\to 0} (2x + h) = 2x
\end{aligned}$$

- So $d(x^2) = f'(x)dx = 2x\,dx$
- Note why this is true: the h in the numerator of the limit corresponds to the dx in the first calculation (in blue)

The differential: two definitions

Suppose that $y = f(x)$ is a polynomial function. The differential of y can be defined either by:

- $dy = f(x + dx) - f(x)$, where we expand everything out, setting $dx^2 = 0$.
- $dy = f'(x)\,dx$, where $f'(x)$ is the derivative of the function $f(x)$.

Procedure for computing derivatives

Suppose that $y = f(x)$. Then $f'(x) = dy/dx$. That is, to find $f'(x)$, we first find dy, then divide it by dx.

The power rule

- $d(x^2) = 2x\,dx$
- $d(x^3) = 3x^2\,dx$

Power rule

If $y = x^p$, then $dy = px^{p-1}\,dx$.

- Let $f(x) = x^2$. Compute $f'(4)$. (Remember that $f'(x) = dy/dx$.)
  With $y = x^2$, $dy = 2x\,dx$, so $f'(x) = \frac{dy}{dx} = 2x$.
  Thus $f'(4) = 2(4) = 8$.
- Let $f(x) = \sqrt{x}$. Compute $f'(4)$.
  $y = \sqrt{x} = x^{1/2}$, so $dy = \frac{1}{2}x^{\frac{1}{2}-1}\,dx$. So $f'(x) = \frac{1}{2\sqrt{x}}$.
  So $f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$.

Idea of algebraic proof

Power rule

If $y = x^p$, then $dy = px^{p-1}\,dx$.

$p = 1$: if $y = x$, then $dy = dx$.
$p = 2$:

$$(x+dx)(x+dx) = (x+dx)(x+dx) = x^2+(x+dx)(x+dx) = x^2+x\,dx+($$

Key point: There are two ways to get the product $x\,dx$, hence the factor of 2

- Similarly, with $(x+dx)^3$, there are three ways to get a dx term:

$$(x+dx)^3 = (x+dx)(x+dx)(x+dx)(x+dx)^3 = (x+dx)(x+dx)(x+$$

- So $d(x^3) = 3x^2\,dx$
- In general, when computing $(x+dx)^n$, there are n ways to get a dx term, so $(x+dx)^n = x^n + nx^{n-1}dx + h.o.t.$

The binomial theorem

Pascal's triangle of binomial coefficients:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 & 0 \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 \end{bmatrix}_x$$

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$x + dx$

$x^2 + 2xdx + dx^2$

$x^3 + 3x^2dx + 3xdx^2 + dx^3$

$x^4 + 4x^3dx + 6x^2dx^2 + 4xdx^3 + dx^4$

$x^5 + 5x^4dx + 10x^3dx^2 + 10x^2dx^3 + 5xdx^4 + dx^5$

$x^6 + 6x^5dx + 15x^4dx^2 + 20x^3dx^3 + 15x^2dx^4 + 6x$

Construction of Pascal's triangle

Proof for positive rational exponents

Power rule

If $y = x^p$ with $p = n/m$, then $dy = px^{p-1}\,dx$.

Proof.

From $y = x^{n/m}$, we have $y^m = x^n$. So $(y + dy)^m = (x + dx)^n$. Expanding both sides using the binomial theorem,

$$y^m + m\,y^{m-1}dy + h.o.t. = x^n + nx^{n-1}\,dx + h.o.t.$$

where $h.o.t.$ means terms involving higher powers of dx and dy. Taking these to be zero, and imposing $y^m = x^n$ gives

$$my^{m-1}\,dy = nx^{n-1}\,dx$$

Again using $y^m = x^n$, we may cancel this common factor, giving

$$my^{-1}dy = nx^{-1}dx \implies dy = \frac{n}{m}yx^{-1}dx = px^px^{-1}dx = px^{p-1}dx$$