Lecture 32: Definite integrals by geometry (minimodule). Two applications

Jonathan Holland

Rochester Institute of Technology*

November 28, 2018

*These slides may incorporate material from Hughes-Hallet, et al, “Calculus”, Wiley

Preview

Skills:

- Evaluate a definite integral of a function whose graph is a line or semicircle using geometry.
- Compute the average value of a function on the interval $[a, b]$ using the formula $\mathrm{Avg}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx$
- To integrate an absolute value, first find where the integrand is zero. These points divide up the domain into intervals on which the integrand is positive or negative. Find the integral in each of those intervals by the FTC.
- The area of the region enclosed between two graphs $y = f(x)$ and $y = g(x)$ on an interval $[a, b]$ is the integral of $|f(x) - g(x)|$.

Interpreting definite integrals as areas

- Sometimes the best way to calculate a definite integral is actually to interpret it as an area.
- Example: $\int_0^3 |x-1|\,dx$. The area under the graph consists of two triangles: one with base from $x=0$ to 1, and height 1, and the other with base from $x=1$ to $x=3$ and height 2. The total area is $1/2+(2)(2)/2=5/2$, which is the value of the integral.
- We could check by dividing the integral into two regions based on where the zeros of $x-1$ are, computing the integral using the FTC in each of the two regions, and finally adding them up (cocycle property).
- Example: $\int_{-2}^2 \sqrt{4-x^2}\,dx$ The graph $y=\sqrt{4-x^2}$ is ... a semicircle! (of radius 2)
- So $\int_{-2}^2 \sqrt{4-x^2}\,dx = \pi r^2/2 = \pi(2)^2/2 = 2\pi$

Average value of a function

- The average of the function $f(x)$ on the interval $[a, b]$ is
$$\mathrm{Avg}(f) = \frac{1}{b-a} \int_a^b f(x)\, dx.$$
- Example: A number is picked at random from the interval $[a, b]$. What is the expected value of that number? The midpoint! Proof:
$$\mathrm{Avg}(f) = \frac{1}{b-a} \int_a^b x\, dx = \frac{1}{b-a}(b^2/2 - a^2/2) = \frac{b+a}{2}$$
by difference of squares.

Expected value of the larger of two numbers

- Let y be in the interval $[0, 1]$, and

$$f_y(x) = \begin{cases} y & \text{if } x < y \\ x & \text{if } x \geq y \end{cases}.$$

- The expected value of f_y is

$$g(y) = \mathrm{Avg}(f_y) = \int_0^1 f_y(x)\,dx = y^2 + (1 - y^2)/2 = (1 + y^2)/2$$

- Now, suppose that the value of y is also selected at random. The expected value of g is

$$\mathrm{Avg}(g) = \int_0^1 (1 + y^2)/2\,dy = 2/3.$$

Integrating an absolute value

- Example: $\int_{-1}^{1} |x| \, dx$
- Example: $\int_{-1}^{2} |x(x-2)| \, dx$

Area between two curves

- Example: Area of petal-shaped region between the curves $y = x$ and $y = x^2$.
- $\int_0^1 |x^2 - x|\,dx$
- $= \int_0^1 (x - x^2)\,dx$ (because on the interval $[0,1]$, x is the top graph and x^2 is the bottom)