Lecture 7: The limit laws

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September 12, 2018

* These slides may incorporate material from Hughes-Hallet, et al, "Calculus", Wiley

Preview

- Terms: **limit**
- Concepts:
  - The (informal) definition of a limit: the *limit* of the function $f(x)$ as x approaches c, written $\lim_{x \to c} f(x)$, to be a number L such that $f(x)$ is as close to L as we want whenever x is sufficiently close to c. If L exists, we write

$$\lim_{x \to c} f(x) = L.$$

- Skills:
  - Use the **limit laws** to evaluate limits
  - Evaluate limits by considering the one-sided limits
  - Use the **squeeze theorem** to evaluate limits

Limits of rational functions (last time)

- If $f(x)$ is a rational function, then the notation

$$\lim_{x \to a} f(x)$$

  is like a recipe that says “First, simplify $f(x)$ as much as possible, and then plug in $x = a$.”
- Numerical meaning of limit.

The idea of a limit

- Not every function is a rational function, but we can make sense of the idea of a limit for other kinds of functions as well.
- We will write $\lim_{x \to c} f(x) = L$ if the values of $f(x)$ approach L as x approaches c.
- How should we find L, or even know whether such a number exists? Look for trends in values in $f(x)$ as x gets closer to c, but $x \neq c$. A graph or table of values is helpful.
- Example: $\lim_{\theta \to 0} \frac{\sin \theta}{\theta}$

Definition of a limit

Definition

A function f is defined on an interval around c, except perhaps at the point $x = c$. We define the *limit* of the function $f(x)$ as x approaches c, written $\lim_{x \to c} f(x)$, to be a number L (if one exists) such that $f(x)$ is as close to L as we want whenever x is sufficiently close to c (but $x \neq c$). If L exists, we write

$$L = \lim_{x \to c} f(x) = L.$$

- Example: Graph $\sin\theta/\theta$ to find out how close θ needs to be to 0 so that $\sin\theta/\theta$ is within 0.01 of 1.

Examples

- $\lim_{x \to 3} \dfrac{x^2 + 5x}{x + 9} = \dfrac{\lim_{x \to 3}(x^2 + 5x)}{\lim_{x \to 3}(x + 9)} =$
  $\dfrac{(\lim_{x \to 3} x)^2 + 5 \lim_{x \to 3} x)}{\lim_{x \to 3} x + 9} = \dfrac{3^2 + 5(3)}{3 + 9}$
- $\lim_{x \to 2} \dfrac{x - 2}{x^2 - 2x} = \lim_{x \to 2} \dfrac{x - 2}{(x - 2)x} = \lim_{x \to 2} \dfrac{1}{x} = \dfrac{1}{2}$

Theorem

If $f(x)$ is a rational function (or polynomial) and $x = c$ is a regular point, then

$$\lim_{x \to c} f(x) = f(c).$$

Note: When $x = c$ is a *removable singularity*, it is not in the domain of f. That means we need to cancel a common factor before we are able to apply this theorem.

A trick for radicals

- Example: $\lim_{x\to 1} \dfrac{\sqrt{x+8}-3}{x-1}$.
- "Rationalize the *numerator*"!

$$\begin{aligned}
\ldots &= \lim_{x\to 1} \frac{\sqrt{x+8}-3}{x-1} \cdot \frac{\sqrt{x+8}+3}{\sqrt{x+8}+3} \\
&= \lim_{x\to 1} \frac{(\sqrt{x+8})^2 - 3^2}{(x-1)(\sqrt{x+8}+3)} \\
&= \lim_{x\to 1} \frac{x+8-9}{(x-1)(\sqrt{x+8}+3)} \\
&= \lim_{x\to 1} \frac{x-1}{(x-1)(\sqrt{x+8}+3)} \frac{\cancel{x-1}}{\cancel{(x-1)}(\sqrt{x+8}+3)} \\
&= \lim_{x\to 1} \frac{1}{\sqrt{x+8}+3} = \frac{1}{\sqrt{1+8}+3}
\end{aligned}$$

The last step ("direct substitution") follows from a combination of the limit laws

When limits do not exist

- Explain why $\lim_{x \to 2} \frac{|x-2|}{x-2}$ doesn't exist. Answer: The one-sided limits are $\lim_{x \to 2^-} \frac{|x-2|}{x-2} = -1$ and $\lim_{x \to 2^+} \frac{|x-2|}{x-2} = +1$, which are not the same.
- Explain why $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ doesn't exist. Answer: The function oscillates wildly as x approaches zero, without ever settling down to converge to any particular y value.

One-sided and two-sided limits

Theorem

A limit exists if and only if the left and right hand limits are equal:

- $\lim_{x\to c} f(x) = L$ *means that* $\lim_{x\to c^+} f(x) = L = \lim_{x\to c^-} f(x)$

Example

Compute $\lim_{x\to 0} |x|$. Recall that

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}.$$

So $\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0$ (direct substitution)and $\lim_{x\to 0^-} |x| = \lim_{x\to 0^-}(-x) = 0$. These are both equal to zero, and so by the theorem $\lim_{x\to 0} |x| = 0$.

An example

- $\lim_{x\to 0} x\sin(1/x)$
- It is tempting to apply the "limit laws" to this:

$$\lim_{x\to 0} x\sin(1/x) = (\lim_{x\to 0} x)(\lim_{x\to 0} \sin(1/x)) = 0.$$

- This is wrong, because the second factor $\lim_{x\to 0} \sin(1/x)$ doesn't exist, so the limit laws do not apply. However, it does suggest a valid line of reasoning.
- Look at the graph of the function $f(x) = x\sin(1/x)$
- Notice that the graph is "squeezed" between the graphs of $|x|$ and $-|x|$, which do tend to a limit as $x \to 0$.

The squeeze theorem

Theorem

If

1. $f(x) \leq g(x) \leq h(x)$ *when* x *is near* a*, and*
2. $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$

then

$$\lim_{x \to a} g(x) = L.$$

Example

We have

$$-|x| \leq x \sin(1/x) \leq |x|.$$

Let $f(x) = -|x|$, $g(x) = x \sin(1/x)$, and $h(x) = |x|$. Then (1) $f(x) \leq g(x) \leq h(x)$ and (2) $\lim_{x \to 0} f(x) = \lim_{x \to 0} h(x) = 0$, so $\lim_{x \to 0} g(x) = 0$.