# 30. The definite integral and net change theorem

## The net change theorem

1. Suppose $q(t)$ is the number of calls received per hour, $t$ hours after a company's customer service center opens at 8:00 am each day. Use the definite integral to write an equation that says, "The customer service center received 205 calls between 9 am and noon."

2. It's often helpful to stratify a population based on age—newborns, toddlers, ..., senior citizens, centenarians—because diseases (among other things) affect them differently. Suppose the population of a city is $\int_0^{150} f(a)\ da$ people, where $a$ is measured in years (and $f(a)$ is zero when $a$ is large). What are the units associated with $f(a)$ and what does it mean to us?

3. Suppose the net electric charge on a rod is $\int_0^1 f(x)\ dx$ coulombs, where $x$ is measured in meters. What are the units associated with $f(x)$ and what does it mean to us?

# Properties of the integral

4. Once again, the horses Sweetie and Thunder are racing. Thunder is faster than Sweetie. (Sweetie is more lovable, but that is not relevant for the problem.) Let $S(t)$ be Sweetie's speed, and $T(t)$ Thunder's speed, both measured in meters per second. We know from the properties of integrals that

$$S(t) \leq T(t) \text{ on } [0, 60] \quad \Rightarrow \quad \int_0^{60} S(t)\ dt \leq \int_0^{60} T(t)\ dt.$$

Explain what this means in the context of the running horses.

5. Solve the puzzle: Suppose $3x + 7 = \int_a^x f(t)\ dt$. What is the value of $a$? [Hint: what does this identity become when $x = a$?]

6. Find a value of $T > 0$ so that the net signed area between the graph of $f(x) = 3x^2 - 4$ and the $x$-axis over $[0, T]$ is zero.

## Finding integrals

7. Integration is hard. Differentiation is easy. Often integration can be so hard that the best way to calculate an integral is just to guess the answer, and then verify that it is correct. In this example, you will calculate

$$\int_0^{\ln(2)} xe^x \, dx.$$

   (a) First we need to find an antiderivative of $xe^x$, that is a function $F(x)$ such that $F'(x) = xe^x$. Based on algebraic intuition, we guess that the form of the integral should be a linear function times the exponential function. So, we write down $F(x) = (Ax + B)e^x$. Calculate $F'(x)$, being careful to use the product rule correctly.

   (b) Your answer to (a) can be factored as $F'(x) =$ (a linear function)$e^x$. Set this equal to $xe^x$ and solve for the unknown constants $A$ and $B$.

   (c) Finally, calculate the definite integral $\int_0^{\ln(2)} xe^x \, dx$ using the fundamental theorem of calculus.

8. An *exponentially damped* oscillator such as a spring has velocity $v(t)$ of the form $V(t) = e^{-t} \cos t$. Our physical intuition tells us that $v(t)$ will have an antiderivative $x(t)$ that is also exponentially damped, but possibly with a phase shift. So we guess an antiderivative of the form $x(t) = (A \cos t + B \sin t)e^{-t}$ for some undetermined coefficients $A$ and $B$. Determine these constants by solving $x'(t) = v(t)$.

9. Calculate the following integrals using the Fundamental theorem. (You may use the “rules” for integration that you learned today, or the guess-and-check method.)

(a) $\int_0^1 e^x \, dx$

(b) $\int_1^2 (x+1)^2 \, dx$

(c) $\int_1^3 x^{-2} \, dx$

(d) $\int_1^e \frac{1}{x} \, dx$ [Hint: what function has derivative equal to $1/x$?]