Lecture 6: Limits of rational functions

Jonathan Holland

Rochester Institute of Technology*

September 10, 2018

*These slides may incorporate material from Hughes-Hallet, et al, "Calculus", Wiley

- Terms: **regular point**, **singular point**, **removable singularity**, **pole**
- Skills:
  - Given a rational function determine whether a given value $x = c$ is a regular point, pole, or removable singularity algebraically/graphically;
  - Determine the end behavior of a rational function by looking at the terms involving the highest powers of x in the numerator and denominator;
  - Find the limit of a rational function at a removable singularity by cancelling common factors, and then plugging in.

- Consider the function $f(x) = \frac{x^2-1}{x-1}$.
- This is not actually literally defined at $x = 1$.
- But of course we know that its value “should” be 2.
- We use the notation

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$

to express this

Calculating limits

- A general rule is to simplify everything as much as possible
- Then plug in the value
- An example:

$$\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = ...$$

- Maybe we have a function $f(x)$ for which $f(a)$ is not literally defined. In that case "$\lim_{x \to a} f(x)$" is a recipe that says "First, simplify $f(x)$ as much as possible, and then plug in $x = a$."

A more interesting example

- $\lim_{x \to 2} \frac{x^2-5x+6}{x^2-4}$

Fact

To compute a limit of the form

$$\lim_{x \to c} \frac{p(x)}{q(x)}$$

when p, q are rational functions, first factor p and q, then cancel any common factors.

Singular points

A rational function $f(x) = \frac{p(x)}{q(x)}$ is given.

- A value $x = c$ is called a *regular point* if $q(c) \neq 0$.
- A value $x = c$ is called a *singular point* (or *singularity*) if $q(c) = 0$.
- If $q(c) = 0$ and $p(c) \neq 0$, c is called a *pole*.
- If $p(x)$ and $q(x)$ have a common factor of $(x - c)^k$, and the singularity goes away when that common factor is cancelled out, then c is called a *removable singularity*.

Example

Let

$$f(x) = \frac{(x^2 - 2x + 1)(x - 2)}{(x^2 - 1)(x^2 - 4x + 4)}$$

Then:

- $x = 1$ is a removable singularity
- $x = -1$ is a pole
- $x = 2$ is also a pole.
- $x = 0$ is a regular point.

One- and two-sided limits

- $\lim_{x \to 2} f(x)$ means the number that $f(x)$ approaches as x approaches 2 from both sides. That is, as x approaches 2 through values greater than 2 (e.g., $2.1, 2.01, 2.003$) and values less than 2 (e.g., $1.9, 1.99, 1.996$).
- If we want x only to approach 2 through values greater than 2, we write $\lim_{x \to 2^+} f(x)$. (A *right-hand limit*)
- If we want x only to approach 2 through values less than 2, we write $\lim_{x \to 2^-} f(x)$. (A *left-hand limit*)

Infinite limits

- $\lim_{x\to 0} \frac{1}{x^2} = +\infty$ means that as x gets closer to 0, the function $1/x^2$ becomes larger and positive, without bound.
- Similarly, $\lim_{x\to 1^-} \frac{1}{x-1} = -\infty$ and $\lim_{x\to 1^+} \frac{1}{x-1} = +\infty$. We can see this by sketching the graph and seeing that the function approaches the asymptote differently on the two sides.

Limits at infinity

- Sometimes we want to know what happens to $f(x)$ as x gets large. "End behavior"
- Example: $\lim_{x\to\infty} \frac{x^2+1}{4x^2+4x-7}$
- Example: $\lim_{x\to-\infty} \frac{x^2+1}{4x^2+4x-7}$

Fact

Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, $q(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0$*. For large* $|x|$*, the approximation*

$$\frac{p(x)}{q(x)} \approx \frac{a_n x^n}{b_m x^m} = \frac{a_n}{b_m} x^{n-m}$$

holds. So it is the leading terms *that govern the end behavior of rational functions.*