Discovery Module: Precise Definition of the Limit

1. Seeing a niche in the local market, you decide to open a franchise of the *The Function Outlet* that specializes in functions f with specified $\lim_{t\to\infty} f(t)$. The umbrella company has warned you that customers are often skeptical, so be prepared to convince them that a function does what you say it does.

   (a) When a customer says, "I see that you have $f(t) = t^2$ in your inventory, and you claim that it grows without bound. If that's true, I should be able to find a time when its value surpasses 10,000. When is that?" Answer the customer, and include a statement about later times in order to assure him of the quality of your product.

   (b) When a customer says, "The packaging on this function, $f(t) = 4^t$, says that it grows without bound. If that's true, I should be able to find a time when its value surpasses 1024. When is that?" Answer the customer, and include a statement about later times in order to assure her of the quality of your product.

   (c) When a customer says, "I found $f(t) = t + 6\cos(10\pi t)$ on the shelf that says *Grows Without Bound.* I need a function that will eventual stay larger than 157. Will this product work for me?" Show the customer that this function will suit his needs (eventually and forever after).

   (d) Generally speaking, if you have a function f in your inventory, and a customer asks whether/when it will surpass a value of L, what must you do?

   (e) Use your answer from part (d) to explain what we mean when we write $\lim_{t\to\infty} f(t) = \infty$. Then draw a heuristic diagram that clearly illustrates what you've written.

   Answer: (a) If $t > 100$, then $f(t) > 10{,}000$

   (b) If $t > 5$, then $f(t) > 1024$

   (c) Note that $\cos(...) > -1$, so $f(t) > t - 6$. Consequently $f(t) > 157$ for all $t > 163$

   (d) We have to find a value of time T, so that $f(t) > L$ for all later values of time, $t > T$. (e) $\lim_{t\to\infty} f(t) = \infty$ means that for each L there is a value of time T such that $f(t) > L$ for all later values of time, $t > T$.

   Note: Said in english, $\lim_{t\to\infty} f(t) = \infty$ when each number L is surpassed by $f(t)$ eventually. The number T plays the role of the word "eventually," making it quantitative.

2. In order to draw more patrons into *The Function Outlet*, you decide to rent space to a coffee house and host live music on Thursday nights. Next to the register where customers place their orders, you put impulse-buys.

   (a) A customer says, "I see that $f(t) = 3 - 1/(4t)$ is in the basket marked *Limit Value 3.* How do you know?" Without looking up from the espresso machine, one of the baristas answers, "The value is never actually 3, but that particular function comes from a family in South America that has a small orchard of *árboles de tresfruta*, which make the best functions for this kind of limit. The family uses their llamas to identify which functions are ripe, and harvests them in late September each year." Seeing the customer start to put down the function, you step up and assure her, "Although it doesn't give you a value of exactly 3, but that function will get as close as you want, eventually." She hesitates, feels the weight of the function in her palm, and asks, "Really? When is it within 10^{-6} of its limit value?" Then she turns to you and adds, "And if it happens at all, what happens after that?" You know your product is a good one. Answer the customer with specifics.

(b) A customer says, "I see that $f(t) = 5 + {}^{10}/_{(2t+7)}$ is in the bin marked *Limit Value 5,* but this function value is clearly larger than 5 at all positive values of t." Having heard your previous conversation, that same barista answers, "The value is never 5, but the function value will get as close to 5 as you want, eventually." The customer looks skeptically at the barista and says, "Oh really? When is it within $^1/_{10}$ of 5?" Then the customer looks to you and adds, "And what happens after that?" You know your product is a good one. Answer the customer with specifics.

(c) Generally speaking, if a function f is in the bin marked as *Limit Value* L, and a customer asks whether/when the function's value comes within ε of L, what must you do?

(d) Use your answer from part (c) to explain what we mean when we write $\lim_{t\to\infty} f(t) = L$. Then draw a heuristic diagram that clearly illustrates what you've written.

Answer: (a) If $t > 250{,}000$ then $f(t)$ is within 10^{-6} of the limit value

(b) If $t > 46.5$, then $f(t)$ is within $^1/_{10}$ of the limit value

(c) ...find a value of time T, so that $f(t)$ is within ε of the limit value for all later values of time, $t > T$.

(d) $\lim_{t\to\infty} f(t) = L$ means that for each $\varepsilon > 0$ there is an associated T such that $|f(t) - L| < \varepsilon$ for all $t > T$.

Note: In part (d), you can think of ε as a "resolution," meaning your ability to distinguish between two numbers. With this interpretation, you can say that $\lim_{t\to\infty} f(t) = L$ when, for any camera with any fixed resolution, the graph of $f(t)$ will eventually get so close to $y = L$ that you will be unable to tell them apart. The number T quantifies the term "eventually."

3. Being an entrepreneur deluxe, you recognize that although your stores are serving the needs of customers who care about the long term behavior of a function, there is also a market for those who care about limit values at finite positions, $x = c$, so you open a new store. But because you don't have the capital for a brick-and-mortar presence, you start your store on-line.

(a) A customer orders a function for which $\lim_{x\to 2} f(x) = 11$, and you supply him with $f(x) = 21 - 5x$. After the product arrives, he calls your customer service line because he wants the function value to be within 0.1 of the claimed value, but when he checks at $x = 1$ and at $x = 3$ he finds that the function values are nowhere near 11. He's angry. Your customer service representative tells the man that he's simply checking the value of f too far away from $x = 2$. "Okay," the man says, "how close to $x = 2$ should I check?" At this question, the employee is stumped and looks to you. Provide an answer.

(b) Another call comes in to the call center later in the day, and you take the call because the other help center representatives are busy. Your company sold the function $f(x) = 4 + 25x$ to a woman who wanted $\lim_{x\to 3} f(x) = 79$. She called the help line because after computing $f(2) = 54$ and $f(4) = 104$, which are very far from 79, she thinks the function was probably damaged in the mail. You explain that if the customer uses her computer to check f close enough to $x = 3$, she won't be able to tell the difference between $f(x)$ and

79. After a few questions, you find out that the resolution of the customer's computer screen allows her to see function values as distinct if they differ by more than $\varepsilon = 0.1$, but it will display them as the same if the difference is less than ε. The customer asks, "With that screen resolution, how close to $x = 3$ do I need to be before the function value looks like 79 to me?" Answer the customer. What will happen if she checks closer than that?

(c) Generally speaking, if a customer calls about a function f that was sold as $\lim_{x\to c} f(x) = L$, and a customer wants their function values to be within ε of L, what must you do to help?

(d) Use your answer from part (c) to write a definition of what we mean when we write $\lim_{x\to c} f(x) = L$. Then draw a heuristic diagram that clearly illustrates what you've written.

Answers: (a) $\delta = 0.02$

(b) $\delta = 0.004$

(c) ... find a distance $\delta > 0$ so that $f(x)$ differs from L by less than ε when x is within δ units of c.

(d) $\lim_{x\to c} f(x) = L$ means that for each $\varepsilon > 0$ there is an associated $\delta > 0$ such that $0 < |x - c| < \delta \Rightarrow \left|f(x) - L\right| < \varepsilon$.

Note: In part (d), you can think of ε as a "resolution," meaning your ability to distinguish between two numbers. With this interpretation, you can interpret $\lim_{x\to c} f(x) = L$ as saying that any "camera" with any fixed resolution will be unable to distinguish between the graph of $f(x)$ and the line $y = L$ when x is "close enough" to c. The number δ quantifies the term "close enough."

Skill Development: In #3 you answered the question "how close?" This required you to make the idea of "close enough" precise by calculating a distance (which depended on the customer's resolution or tolerance for error). Generally speaking, we denote the distance "close enough" by the symbol δ. The exercises below will provide you with practice showing that $\lim_{x\to c} f(x) = L$ with various c, L, and f by computing δ for various values of ε, and establishing the relationship in general.

4. Suppose your computer screen is 1920 pixels (width) by 1080 pixels (height), and you use a graphing program to display the graph of $f(x) = \frac{6x+1}{3x+21}$ in the window $-8 \le x \le 8$ and $-4.5 \le y \le 4.5$. For the sake of simplicity, let's assume that the graphing window takes up the entire computer screen. (Questions on the next page.)

   (a) Because there are 1080 pixels in each column of the screen, each pixel corresponds to a small range of y-values. We'll denote the size of this small vertical increment by ε. Given that y ranges over $[-4.5, 4.5]$, what is the value of ε?

   (b) Suppose you also graph the line $y = 2$ on the same screen. Because $\lim_{x\to\infty} f(x) = 2$, the graph of f will approach the horizontal line. If you shift the viewing window over by T, so that it displays $-8 + T \le x \le 8 + T$, but maintain $-4.5 \le y \le 4.5$, how far to you have to shift it so that the screen can no longer resolve the difference in height between the curve $y = f(x)$ and the line $y = 2$?

   Answer: (a) $\varepsilon = {}^{9}/_{1080}$

   (b) $T = 1633$

5. The limit $\lim_{x\to c} f(x)$ is the number $L = 10$ when $f(x) = 2x + 4$ and $c = 3$.

   (a) If $\varepsilon = 1$, what is δ?

   (b) If $\varepsilon = 0.25$, what is δ?

   (c) If $\varepsilon > 0$ is specified, what formula gives us δ?

   Answer: (a) $\delta = 0.5$

   (b) $\delta = 0.125$

   (c) $\delta = {}^{\varepsilon}/_{2}$

6. Repeat #5 with $L = 2$, $c = 0$, and

$$f(x) = \begin{cases} 3x + 2 & \text{if } x \ge 0 \\ -5x + 2 & \text{if } x < 0 \end{cases}$$

   Answer: (a) $\delta = 0.2$

   (b) $\delta = 0.05$

   (c) $\delta = {}^{\varepsilon}/_{5}$

7. Repeat #5 with $L = 16$, $c = 4$, and $f(x) = x^2$. Answer: (a) $\delta = \sqrt{17} - 4$

(b) $\delta = \sqrt{16.5} - 4$

(c) $\delta = \sqrt{16 + \varepsilon} - 4$