MATH 181A, Workshop 26: Exam 2 review

1. In this problem, let

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Each part is 4 points.

(a) Compute $f'(x)$ for $x \neq 0$.
$f'(x) = -\cos(1/x)/x^2$

(b) Decide if $f(x)$ is differentiable at $x = 0$. If so, find its derivative.
Not differentiable.

(c) Find the equation of the tangent line to $y = f(x)$ at the point $(4/\pi, 1/\sqrt{2})$.
$f'(4/\pi) = -\frac{\pi^2}{16\sqrt{2}}$, so the equation is

$$y - 1/\sqrt{2} = -\frac{\pi^2}{16\sqrt{2}}(x - 4/\pi)$$

(d) Compute $\lim\limits_{x\to 0} xf(x)$ and $\lim\limits_{x\to\infty} xf(x)$. [Hint: Only one of these is an indeterminate form.]
We have $\lim_{x\to 0} xf(x) = 0$ by the squeeze theorem. By L'Hopital,

$$\begin{aligned} \lim_{x\to\infty} xf(x) &= \lim_{x\to\infty} \frac{\sin(1/x)}{1/x} \\ &= \lim_{x\to\infty} \frac{-\cos(1/x)/x^2}{-1/x^2} \\ &= \lim_{x\to\infty} \cos(1/x) = 1 \end{aligned}$$

2. Find the equation of the tangent line to the curve $-x^3 + 2y^2 + xy = 8$ through the point $(-2, 1)$.

We have $-3x^2dx + 4ydy + xdy + ydx = 0$. At the point $(-2, 1)$, with $dx = x + 2$ and $dy = y - 1$, the equation of the tangent line is

$$-12(x + 2) + 4(y - 1) - 2(y - 1) + (x + 2) = 0$$

6 points: Implicit differentiation 2 points: Evaluating at the point 4 points: Correct tangent line

3. Find the critical points and inflection points of $f(x) = x^5 + 15x^4 + 25$. Classify the critical points as local maxima, local minima, or neither.

$f'(x) = 5x^4 + 60x^3 = 5x^3(x+12) = 0$ implies $x = 0$ or $x = -12$

If $x < -12$, then $f'(x) > 0$. If $-12 < x < 0$, then $f'(x) < 0$. If $0 < x$, then $f'(x) > 0$.

So $x = -12$ is a local max, $x = 0$ is a local min.

$f''(x) = 20x^3 + 180x^2 = 20x^2(x+9) = 0$ implies $x = 0$ or $x = -9$

If $x < -9$, $f''(x) < 0$ so f is concave down. If $-9 < x < 0$, then $f''(x) > 0$, so f is concave up. If $x > 0$, then $f''(x) > 0$, so f is also concave up. Thus $x = -9$ is the only inflection point.

4. Compute the following limits.

(a) $\lim\limits_{x\to\infty} \frac{6x^2+1}{11x^2+2x+1}$

$6/11$

(b) $\lim\limits_{x\to\pi} \frac{\sin(2x-2\pi)}{x^2-\pi^2}$

By L'Hopital,

$\lim\limits_{x\to\pi} \frac{\sin(2x-2\pi)}{x^2-\pi^2} = \lim\limits_{x\to\pi} \frac{2\cos(2x-2\pi)}{2x} = \frac{1}{2\pi}$

(c) $\lim\limits_{x\to 0} \frac{1-\cos(\pi x)}{x^2}$

By L'Hopital

$\lim\limits_{x\to 0} \frac{1-\cos(\pi x)}{x^2} = \lim\limits_{x\to 0} \frac{\pi\sin(\pi x)}{2x} = \lim\limits_{x\to 0} \frac{\pi^2\cos(\pi x)}{2} = \frac{\pi^2}{2}$

(d) $\lim\limits_{x\to 0^+} [\sin(x)\cdot\ln(x)]$

By L'Hopital

$\lim\limits_{x\to 0^+} [\sin(x)\cdot\ln(x)] = \lim\limits_{x\to 0^+} \frac{\ln(x)}{1/\sin x} = \lim\limits_{x\to 0^+} \frac{1/x}{-\cos x/\sin^2 x} = \lim\limits_{x\to 0^+} \frac{\sin^2 x}{-x\cos x} = \lim\limits_{x\to 0^+} \frac{2\cos x\sin x}{-\cos x + x\sin x} =$
0

5. Let $f(x) = x^4 - 8x^2$ for $-3 \le x \le 1$. Find the global maximum and minimum for the function.

At the stationary points, $f'(x) = 4x(x^2-4) = 0$. So $x = -2, 0, 2$. Of these, the last one is not in the domain.

x	$f(x)$
-3	9
-2	-16
0	0
1	-7

So $f(-3) = 9$ is tha absolute max, and $f(-2) = -16$ is the absolute min.

6. Compute the derivatives.

(a) $\dfrac{d}{dx}\dfrac{e^x}{x+1}$ Ans: $\frac{e^x(x+1)-e^x}{(x+1)^2}$

(b) $\dfrac{d}{dx}\tan(2x)$ Ans: $2\sec^2(2x)$

(c) $\dfrac{d}{dx}\arctan(x/\pi)$ Ans: $\frac{\pi^{-1}}{1+(x/\pi)^2}$

(d) $\dfrac{d}{dx}\left[e^x\sin(x)\right]$ Ans: $e^x(\sin x+\cos x)$

(e) $\dfrac{d}{dx}\left(x^3+2x^{-1}+\sqrt{x}\right)$ Ans: $3x^2-2x^{-2}+1/(2\sqrt{x})$

(f) $\dfrac{d}{dx}\ln(x^2+1)$ Ans: $2x/(x^2+1)$

7. A pumpkin is tossed straight into the air, with an inital velocity of $50 ft/sec$. The pumpkin is 5 feet above the ground when it is released. Its height, in feet, at time t seconds is given by

$$y = -16t^2 + 50t + 5.$$

(a) What is the velocity of the pumpkin after t seconds?
$dy/dt = -32t + 50$

(b) How high does the pumpkin go before returning to the ground, and exploding into a gooey mess?

At the maximum height, $dy/dt = 0$, so $t = 50/32$.

8. Suppose that x and y are two functions of t such that $y^2 - 2x - 1 = 0$. If $dy/dt = 5$, find dx/dt when $x = 12$.

We have $2y\,dy - 2\,dx = 0$. Dividing through by dt gives the related rate equation

$$2y\frac{dy}{dt} - 2\frac{dx}{dt} = 0$$

or

$$\frac{dx}{dt} = 2y\frac{dy}{dt}.$$

Now $x = 12$, and $y^2 - 2x - 1 = 0$, implies $y^2 = 25$, so $y = \pm 5$. So

$$\frac{dx}{dt} = \pm 50$$

9. You have the following information about a twice-differentiable function $y = f(x)$ on the interval $0 \leq x \leq 10$.

- The critical points of $f(x)$ are the points of the graph $y = f(x)$ with (x, y) coordinates $(2, 0)$, $(4, 2)$, $(6, 0)$, $(8, 4)$
- You calculate the derivative of f at some points: $f'(1) = 7$, $f'(3) = 2$, $f'(5) = -2$, $f'(7) = 12$, $f'(9) = -1$
- Finally, you compute $f(0) = -1$ and $f(10) = 3$.

(a) Locate the global extrema of $f(x)$.
Ans: Global max at $f(8) = 4$. Global min: $f(0) = -1$

(b) Complete the table

	$0 < x < 2$	$2 < x < 4$	$4 < x < 6$	$6 < x < 8$	$8 < x < 10$
sign of f'	$+$	$+$	$-$	$+$	$-$
behavior of f	$\nearrow$	$\nearrow$	$\searrow$	$\nearrow$	$\searrow$

(c) Find all local extrema of f, and classify them as local maxima and local minima. You may neglect the endpoints.
Local maxima at $x = 4, 8$. Local minimum at $x = 6$.

(d) You calculate the second derivative of f, and find where it is zero. It is zero exactly at the points $x = 2, 3.5, 5.5, 7$. Complete the table (there is only one way to do this with the given information).

	$0 < x < 2$	$2 < x < 3.5$	$3.5 < x < 5.5$	$5.5 < x < 7$	$7 < x < 10$
sign of f''	$-$	$+$	$-$	$+$	$-$
behavior of f					

(e) Sketch the graph of $y = f(x)$, showing all critical points, inflection points, intervals of increase and decrease.

10. Here is a table of values of f, g, f', g'.

x	0	1	2	3
$f(x)$	2	1	3	1
$f'(x)$	1	0	2	1
$g(x)$	1	2	3	4
$g'(x)$	4	3	2	1

Let $h(x) = f(g(x))$. Find $h'(2)$.

Ans: $h'(2) = f'(g(2))g'(2) = f'(3)(2) = (1)(2) = 2$

11. One vehicle is heading West toward an intersection at 80 miles per hour while another is heading North away from the intersection at 60 miles per hour. At the moment when the North bound vehicle is 3 miles North of the intersection and the West bound vehicle is 4 miles East of the intersection, determine whether the vehicles are getting closer or further apart, and at what rate the distance between them is changing.

Answer: we have $x^2 + y^2 = z^2$ where x is the distance of the first car to the intersection, y is the distance of the second car, and z is the distance between the two cars. So $2x\,dx + 2y\,dy = 2z\,dz$. Dividing by dt gives the related rate equation

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}.$$

Substituting in $x = 4, y = 3, z = 5$, and $dx/dt = -80$, $dy/dt = +60$, we have

$$2(4)(-80) + 2(3)(60) = 2(5)\frac{dz}{dt}.$$

So $dz/dt = -64 + 36 = -28$. So the vehicles are getting closer together at a rate of 28 miles per hour.

12. Estimate $\sqrt{9.1}$ accurately to two decimal places without the aid of a calculator. [You may use differentials, the tangent line approximation to $y = \sqrt{x}$ at $x = 25$, or any other mathematically rigorous method at your disposal.]

Initial guess: $x_0 = 3$. $x = 3 + dx$. $x^2 = 9 + 6dx = 9.1$, so $dx = 0.1/6 \approx 0.017$, and $x = 3.017$.