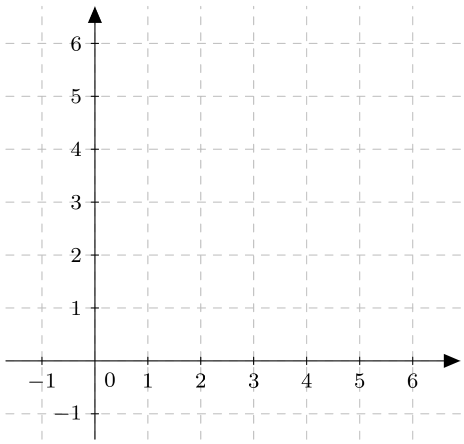
The Cartoon Test and Extreme Value Table

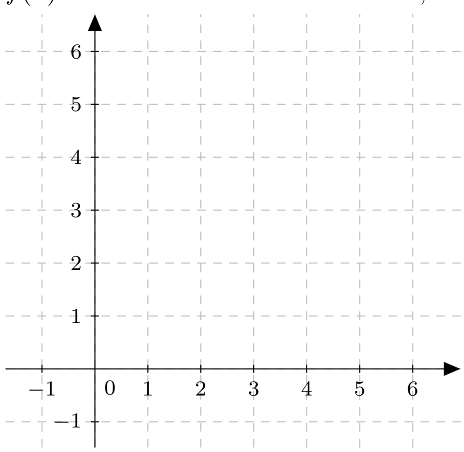
Summary: There are two different table methods for maximization and minimization. One table method allows us to make conclusions about the *local behavior* of the function, and in particular its local maxima and minima. The other table method allows us to draw global conclusions, and in particular locate the absolute maximum and minimum of a function. On a closed and bounded interval, any continuous function has an absolute maximum and an absolute minimum: these may be either at an interior critical point, or at a boundary point of the domain.

1. For each of the following, sketch a graph of a differentiable function $y = f(x)$ on the domain $[0, 5]$ with the indicated property, or say why it is impossible.

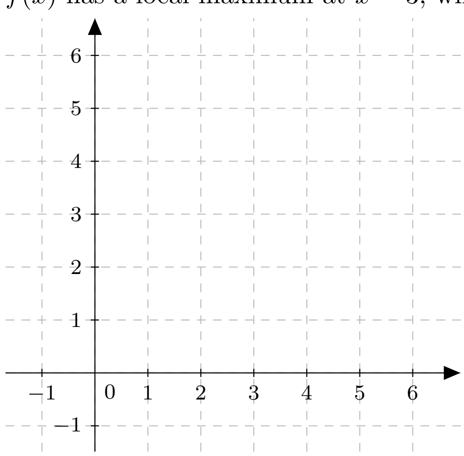
   (a) $f(x)$ has an absolute maximum at $x = 0$ and an absolute minimum at $x = 2$.



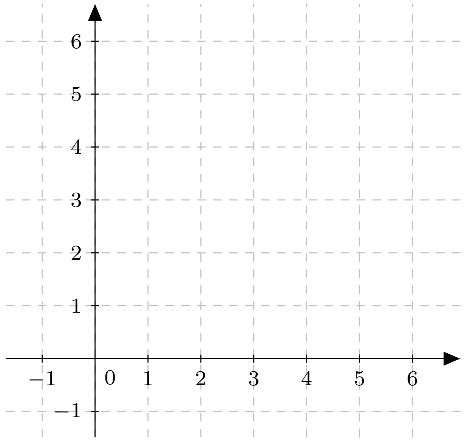
   (b) $f(x)$ has a local maximum at $x = 2$, which is its only stationary point, and an absolute minimum at $x = 5$.



   (c) $f(x)$ has a local maximum at $x = 3$, which is its only stationary point, and an absolute maximum at $x = 5$.



(d) $f(x)$ has a stationary point at $x = 2$, which is neither a local maximum nor a local minimum.



If you have a function $f(x)$ and you want to determine its local extrema, you can complete the following steps. Let us do this for the function $f(x) = \frac{x-1}{x^2+1}$.

- Find the stationary points of f. These are points where $f'(x) = 0$.
  To do this, we need to take the derivative of $f(x)$:

$$\begin{aligned} f'(x) &= \frac{(1)(x^2+1) - (x-1)(2x)}{(x^2+1)^2} && \text{using the quotient rule} \\ &= \frac{1+2x-x^2}{(x^2+1)^2} \end{aligned}$$

  So the stationary points, where $f'(x) = 0$, are where the numerator is zero. So these are the roots of the quadratic $1 + 2x - x^2 = 0$. Applying the quadratic formula gives the stationary points as

$$x = 1 \pm \sqrt{2}.$$

- Draw the points on a number line, and make a table below the number line. Think of these points as marking "dividers" that go between two neighboring columns of the table.

	$x < 1-\sqrt{2}$	$1-\sqrt{2} < x < 1+\sqrt{2}$	$1+\sqrt{2} < x$
Sample point			
Sign of f'			
Behavior of f			

  Note: Each column of the table corresponds to an interval where the derivative $f'(x)$ is either positive or negative, because by a version of the intermediate value theorem, if the derivative were to go from positive to negative, it would need to cross zero somewhere. So assuming we've found all of the zeros correctly, we will be able to determine the sign of the derivative in each of these intervals just by *testing a sample point.*

- Fill in the boxes of the table: to determine whether f' is positive or negative, we just need to test a point in each interval.

	$x < 1-\sqrt{2}$	$1-\sqrt{2} < x < 1+\sqrt{2}$	$1+\sqrt{2} < x$
Sample point	$x = -2$	$x = 1$	$x = 3$
Sign of f'	$f'(-2) = -7/25 \implies \boxed{-}$	$f'(1) = 1/2 \implies \boxed{+}$	$f'(3) = -1/50 \implies \boxed{-}$
Behavior of f	$\searrow$	$\nearrow$	$\searrow$

- Using this table, we can identify the local extrema of f.
  The function f has a local minimum at $x = 1 - \sqrt{2}$ and a local maximum at $x = 1 + \sqrt{2}$.

2. Let $f(x) = \frac{3x-4}{x^2+1}$.

(a) Using the quotient rule, we can find that $f'(x) = \frac{-3x^2+8x+3}{(x^2+1)^2}$. Find the stationary points of f.

(b) Fill in the table.

Sample point			
Sign of f'			
Behavior of f			

(c) Classify the stationary points as local minima, maxima, or neither.

(d) Compute $\lim_{x\to\infty} f(x)$. What is the horizontal asymptote of $f(x)$?

(e) Decide whether $f(x)$ has any absolute maximum or minimum. (Hint: Try to draw a graph. Do not use a calculator!)

3. Consider function $f(x) = (x-1)^3(x-2)$.

(a) What is the local behavior of $f(x)$ near the zero at $x = 1$?

(b) Is $x = 1$ a local maximum, local minimum, or neither?

(c) Find both stationary points of $f(x)$.

(d) Fill in the table.

Sample point			
Sign of f'			
Behavior of f			

(e) Use your table to find all local extrema of $f(x)$.

4. The *extreme value theorem* says that a continuous function on a closed and bounded interval $[a, b]$ has an absolute maximum and an absolute minimum. This is an important theorem in applied mathematics because it is often required to guarantee that there are *optimal solutions* to certain problems.

In concrete terms, this theorem gives us a procedure for finding the absolute extrema of a continuous function $f(x)$ on an interval $[a, b]$:

- The absolute extrema of $f(x)$ must happen either at a stationary point, or at an endpoint of the domain.

So, to find the absolute maximum and minimum of $f(x)$, we simply need to find the stationary points of $f(x)$, and tabulate the value of $f(x)$ at these stationary points and at the endpoints of the domain.

For example, let us investigate the absolute maximum and minimum of the function $f(x) = x^3 - 6x^2 + 9x - 2.$ on the interval $[0, 5]$.

(a) What are the endpoints of the domain of $f(x)$?

(b) Compute $f'(x)$.

(c) Find the stationary points of $f(x)$. [These are the values of x that make $f'(x) = 0$.]

(d) Complete the following table by putting in the first column the values of the stationary points and the endpoints of the domain. Also compute the value of the function $f(x)$ at each of these points.

x	Value of $f(x)$

(e) What is the absolute maximum and absolute minimum of the function $f(x)$ on the domain $[0, 5]$?