Math 181: Exam 2 practice solutions

Name:____

1. Find the equation of the tangent line to the curve $x^2 + y^2 + xy = 7$ through the point $(-3, 1)$.

We have

$$\begin{aligned} d(x^2 + y^2 + xy) &= d(7) \\ 02x\,dx + 2y\,dy + dx\,y + x\,dy &= 0 \\ 2(-3)\,(x+3) + 2(1)\,(y-1) + (x+3)\,(1) + (-3)\,(y-1) &= 0 \end{aligned}$$

2. Find the critical points and inflection points of $f(x) = x^3 - 6x^2 + 9x + 15$. Classify the critical points as local maxima, local minima, or neither. Show correct work, including tables of signs and behavior of function.

$f'(x) = 3(x^2 - 4x + 3) = 3(x-1)(x-3)$

For $x < 1$, $f'(x) > 0$.

For $1 < x < 3$, $f'(x) < 0$.

For $3 < x$, $f'(x) > 0$.

So $x = 1$ is a local max and $x = 3$ is a local min.

3. Let $f(x) = x^3 - 3x^2 - 9x$ for $-2 \le x \le 2$. Find the global maximum and minimum for the function.

We have $f'(x) = 3(x^2 - 2x - 3) = 3(x-3)(x+1)$. The stationary points are 3 and -1, but only one of these lies in the interval -1. Make the table of values of $f(x)$ at the stationary point and the endpoints of the domain:

x	$f(x)$
-2	-2
-1	5
2	-22

so the absolute max is $f(-1) = 5$ and the absolute min is $f(2) = -22$.

4. Compute the derivatives.

(a) $\frac{d}{dx}\frac{e^x}{x+1}$ Ans: $\frac{e^x(x+1)-e^x}{(x+1)^2} = \frac{xe^x}{(x+1)^2}$

(b) $\frac{d}{dx}\tan(2x)$ Ans $2\sec^2(2x)$

(c) Recall the rule $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$. Use this rule to compute $\frac{d}{dx}\arctan(2x/\pi)$ Ans: $\frac{2}{\pi}\cdot\frac{1}{1+(2x/\pi)^2}$

(d) $\frac{d}{dx}\left[e^x \sin(x)\right]$ And: $e^x(\sin x + \cos x)$

(e) $\frac{d}{dx}\left(\sqrt{x}+\frac{1}{x^2}+\ln x\right)$ Ans: $1/(2\sqrt{x}) - \frac{2}{x^3} + \frac{1}{x}$

(f) $\frac{d}{dx}\ln(1-x^3)$ Ans: $-3x^2/(1-x^3)$

5. Hillary flies on her broomstick $600ft$ above the ground at a speed of $50ft/sec$, parallel to the ground, in a direction towards the White House. How fast is her distance to the White House changing when she is $1200ft$ from it?

Let x, y be the horizontal and vertical distance, and z be the distance to the white house. So $z^2 = x^2 + y^2$. Taking the differential gives $2z\,dz = 2x\,dx + 2y\,dy$. Since she flies parallel to the ground, $dy = 0$, and so $2z\,dz = 2x\,dx$. Dividing by dt gives the related rate equation:

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt}.$$

We are given $z = 1200$ and $y = 600$, so $x = 600\sqrt{3}$. Also $dx/dt = 50$. So

$$\frac{dz}{dt} = 25\sqrt{3}$$

6. (a) State the extreme value theorem.

If f is a continuous function on the closed interval $[a, b]$, then f has an absolute maximum and absolute minimum somewhere in the interval.

(b) Draw the graph of a function on the domain $0 \le x \le 1$ that has no absolute maximum, if possible. If it is not possible, explain why not.

It is possible, but the function needs to be discontinuous. Consider the function

$$f(x) = \begin{cases} x & x < 1 \\ 0 & x = 1 \end{cases}.$$

The maximum "should" be 1, but this value is never actually attained.

7. A ladder $3m$ long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of $1m/s$, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is $1m$ from the wall?

The quantities x, y are related by $x^2+y^2 = 9$. Differentiating gives $2x\,dx+2y\,dy = 0$. Dividing by dt gives the related rate equation $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$. Plugging in $x = 1$, $y = 2\sqrt{2}$ and $dx/dt = 1$,

$$2 + 4\sqrt{2}\frac{dy}{dt} = 0$$

so $dy/dt = -\frac{1}{2\sqrt{2}}$.

8. Compute the linearization of the function $y = x^2 + x + 1$ at the point $x = 1$.

$L(x) = f(a) + f'(a)(x - a)$

$f(1) = 3,\ f'(1) = 3$. So $L(x) = 3 + 3(x - 1)$.

9. Use differentials to find a solution x to the equation $x^3 = 27.27$ that is accurate to two decimal places. [You must show valid steps.]

Choose the initial guess $x = 3$. Then add a small correction $x = 3 + dx$. We have

$$x^3 = 27 + 3(3)^2\,dx = 27.27$$

so $dx = 0.01$. Our improved guess is therefore $x = 3.01$.

10. Ohm's law states that the voltage V applied to a resistor of R ohms (a unit of resistance) is

$$V = IR$$

where I is the current in amperes. Assume that V is constant. The resistance of a resistor is determined experimentally by measuring the current that an applied voltage produces. Find the relationship between the relative error dI/I in the measured value of the current and the relative error dR/R in the computed value of the resistance.

By the product rule

$$dV = dI\,R + I\,dR.$$

Divide through by $V = IR$ to get

$$\frac{dV}{V} = \frac{dI}{I} + \frac{dR}{R}$$

Since the voltage is constant $dV = 0$, and we get the relationship

$$\frac{dI}{I} = -\frac{dR}{R}.$$