13. Review for exam 1

1. Find all solutions x to the equation $2^{x^2} = 3^{x+1}$. You may leave your answer in terms of natural logarithms.
   Solution. Taking logarithms gives $x^2 \ln 2 = (x+1)\ln 3$, so $x^2 \ln 2 - x \ln 3 - \ln 3 =$. Appling the quadratic formula gives...

2. Compute the limits:

   (a) $\displaystyle\lim_{x\to\infty} \frac{x-2}{x^2-1}$

   (b) $\displaystyle\lim_{x\to 1^+} \frac{x-2}{x^2-1}$

   (c) $\displaystyle\lim_{x\to\infty} \frac{3x^2-12}{x^2-3x+2}$

   (d) $\displaystyle\lim_{x\to 2} \frac{3x^2-12}{x^2-3x+2}$

   (e) $\displaystyle\lim_{x\to\sqrt{2}} \frac{3-\sqrt{5x^2-1}}{x^2-2}$

   (f) $\displaystyle\lim_{x\to\infty} \frac{1-\sqrt{4x^2+1}}{3x-2}$

   Answers: (a) 0, (b) $-\infty$, (c) 3, (d) 12, (e) $-5/6$, (f) $-2/3$

3. This problem concerns the function $f(x) = \frac{x+1}{x+2}$.

   (a) Find the *domain of* $f(x)$.
   All $x \neq -2$

   (b) Find any *vertical asymptotes* of $y = f(x)$.
   There is a VA at $x = -2$

   (c) Find any *horizontal asymptotes* of $y = f(x)$.
   There is an HA at $y = 1$

   (d) Compute $\displaystyle\lim_{x\to -2^+} \frac{x+1}{x+2}$.

   Near $x = -2$, we have $\frac{x+1}{x+2} \approx \frac{-2+1}{x+2} = \frac{-1}{x+2}$. Since this is negative to the right of its asymptote, $\displaystyle\lim_{x\to -2^+} \frac{x+1}{x+2} = -\infty$.

   (e) Compute $\displaystyle\lim_{x\to -2^-} \frac{x+1}{x+2}$.
   $\displaystyle\lim_{x\to -2^-} \frac{x+1}{x+2} = +\infty$.

   (f) Compute $\displaystyle\lim_{x\to\infty} \frac{x+1}{x+2}$.
   $\displaystyle\lim_{x\to\infty} \frac{x+1}{x+2} = 1$.

   (g) Find the inverse function of $f(x)$.
   We want to solve $y = \frac{x+1}{x+2}$ for x in terms of y. Clearing denominators gives $y(x+2) = x+1$. Now, we want to move all terms involving x to one side of the equation, and all terms not involving x to the other side. We have $yx - x = -2y + 1$. Factoring the left-hand side gives $x(y-1) = -2y+1$ so $x = \frac{y-1}{-2y+1}$. Thus $f^{-1}(x) = \frac{x-1}{-2x+1}$

   (h) Find the range of $f(x)$.
   All $x \neq 1$.

4. Consider the polynomial $f(x) = x^2(x-2)^3(x-1)(x+1)^2$. Determine the local behavior of $f(x)$ near each of the zeros $x = -1, 0, 1, 2$. Sketch a graph of the function, showing the correct end behavior.

   Answer:

$$\begin{aligned} \text{Near } x = -1, &\quad f(x) \approx 3^3 \cdot 2 \cdot x^2 \\ \text{Near } x = 0, &\quad f(x) \approx 2x^2 \\ \text{Near } x = 1, &\quad f(x) \approx -4(x-1) \\ \text{Near } x = 2, &\quad f(x) \approx 72(x-2)^3 \end{aligned}$$

5. In 2006, the population of Iceland was 300,000. In 2016, the population is 330,000. Assume that the population grows exponentially.

   (a) Write the exponential model $P = P_0 a^t$ that best fits this data. (The variable t can be in years from 2006.)

   (b) Use your model to estimate the population of Iceland in 2026.

   (c) Find how long it will take the population to double.

   In units of thousands, $P = 300(1.1)^{t/10}$. So $P(2) = 300(1.1)^2 = 300(1.21) = 363$ thousand. To find the doubling time, we want to solve $(1.1)^{t/10} = 2$ for t. Logarithms gives $\frac{t}{10} \ln 1.1 = \ln 2$ so $t = \frac{10 \ln 2}{\ln 1.1}$

6. (a) State the intermediate value theorem.

   If $f(x)$ is a continuous function for $a \leq x \leq b$, then whenever k is a value between $f(a)$ and $f(b)$, there is a value c of x between a and b for which $f(c) = k$.

   (b) The function $f(x) = x^3 + \sin x$ is known to be continuous. Prove that a solution of the equation $f(x) = 1$ exists.

   We have $f(0) = 0 < 1$ and $f(\pi/2) = \pi^3/8 + 1 > 1$, so there is a value of x between 0 and $\pi/2$ where $f(x) = 1$.

7. Give an example of a rational function $f(x)$ satisfying *all* of the following conditions:

   - $\lim_{x\to\infty} f(x) = 1$
   - $\lim_{x\to 1} f(x) = -\infty$
   - $f(0)$ is undefined, but $\lim_{x\to 0} f(x) = 2$.

     An example is

$$f(x) = -\frac{x}{x(x-1)^2} + \frac{x^2+3}{x^2+1}$$