Math 181, Fall 2018 Handout: The rules for differentials

The rules for differentials. u and v denote any expressions, made out of atomic variables x, y, z, constant numbers like $e, 1, 2, \pi, i, \mathtt{a}$, etc, arithmetic operations of addition, division, multiplication, exponentiation, and unary transcendental functions $(\sin, \cos, \tan, \arctan, \arcsin, \ln)$ Also $\mathtt{a}$ is a constant. The differential d, is an operator that takes expressions to expressions, and satisfies the following Laws:

Law 1. $d(u^v) = u^v \left(\ln(u) d(v) + \frac{v}{u} d(u)\right)$

- In the special case when $v = \mathtt{a}$ is constant, $d(u^{\mathtt{a}}) = \mathtt{a} u^{\mathtt{a}-1} d(u)$
- Also, we have the special case $d(e^u) = e^u du$.

Law 2. $d(\mathtt{a} \cdot u) = \mathtt{a} \cdot d(u)$

Law 3. $d(u + v) = d(u) + d(v)$

Law 4. $d(\mathtt{a}) = 0$

Law 5. $d(uv) = v\, d(u) + u\, d(v)$

Law 6. $d\left(\dfrac{u}{v}\right) = \dfrac{v\, d(u) - u\, d(v)}{v^2}$

Law 7. $d(\sin u) = (\cos u)\ d(u)\,, \qquad d(\cos u) = -(\sin u)\ d(u)$

Law 8. $d(\ln u) = \frac{d(u)}{u}$

Law 9. $d(\arctan u) = \dfrac{d(u)}{1 + u^2}, \qquad d(\arcsin u) = \dfrac{d(u)}{\sqrt{1 - u^2}}$

Once d reaches an atomic variable, it does not simplify further: e.g., $d(x) = dx$. We say in that case that the bird has found the food, and we leave the bird together with its food as a single bird-with-food symbol, dx, dy, etc.

For example,

$$\begin{aligned}
d(x^3 + 4)^{27} &= 27(x^3 + 4)^{26} d(x^3 + 4) && \text{Law 1 with } u = x^3 + 4 \text{ and } \mathtt{a} = 27 \\
&= 27(x^3 + 4)^{26}(d(x^3) + d(4)) && \text{Law 3} \\
&= 27(x^3 + 4)^{26} \cdot 3x^2 dx && \text{Law 1 with } u = x^3 \text{ and } \mathtt{a} = 3 \text{ (and Law 4)}
\end{aligned}$$

Use the rules for differentials to simplify the following, so that the bird gets the food. Apply one law per step, and cite each law as you go.

1. $d(x^3)$

2. $d(ye^x)$

3. $d(x^2 + 1)$

4. $d\left(x \cos(x^2 + 1)\right)$

5. $d\left(\frac{\cos(x^2+1)}{(x^3+4)^{27}}\right)$

6. $d\left(x^2 + y^2\right)$