Handout: Optimization (solution)

The goal of an optimization problem is to maximize or minimize some quantity. Optimization problems have two components: the *constraint* and the *objective*. The objective is the quantity we wish to maximize or minimize, and the constraint is an equation that is imposed from the problem. The following example illustrates the role of the constraint and objective:

> A farmer wishes to build a fence to enclose a rectangular $800ft^2$ area outside his barn. One side of the area will be adjacent to the barn, while the other three sides require fencing. If fencing costs \$50 per foot, what is the minimum cost?

Objective. Two parallel sides have lengths y, and the remaining side has length x (the fourth side is adjacent to the barn and does not require fencing). The total cost is $C = 50(x + 2y)$. This is the objective function.

Constraint. The constraint is that the area we need to enclose is $800ft^2$. As an equation, this is $xy = 800$.

We can solve the constraint to eliminate one of the variables: $y = 800/x$. Substituting this back into the objective, $C(x) = 50(x + 2(800/x)) = 50x + 80000/x$. The problem is thus reduced to

> Minimize the cost $C(x) = 50x + 80000/x$, for $x > 0$.

We do this by looking for stationary points and applying the extreme value theorem or the first derivative test. To find the stationary points

$$C'(x) = 50 - 80000/x^2 = 0$$

so $50x^2 = 80000$, which implies $x = 40$. Also, $C''(x) = 160000/x^3 > 0$ at the critical point, so C is concave up, and therefore $x = 40$ is an absolute minimum of $C(x)$. Thus, the dimensions minimizing the cost are $x = 40, y = 20$.

- Indentify the objective and constraint.
- Solve the constraint to reduce the problem to a one variable optimization problem.
- Find the stationary points of the objective function.
- Analyze these stationary points to determine which (if any) solves the problem. (Using the second derivative test for an open domain, or the extreme value theorem for a closed domain.)

1. Consider the following problem:

   > A box with a square base and an open top must have a volume of $32{,}000cm^3$. Find the dimensions of the box that minimize the amount of material used.

   What is the objective function?

   $A = x^2 + 4xy$

   What is the constraint?

   $V = x^2y = 32{,}000$

   Solve the constraint and substitute back into the objective function.

   $y = 32{,}000/x^2$

   So the objective is

   $A = x^2 + 128{,}000/x$

   Find the stationary points of the objective function.

   $A' = 2x - 128{,}000/x^2 = 0$

   So $x^3 = 64{,}000$, $x = 80$.

   Analyze the stationary points.

   If $x < 80$, $A' < 0$. If $x > 80$, $A' > 0$. So $x = 80$ is an absolute minimum.