5. The tangent line and Leibniz approximation

Much of calculus concerns the problem of finding useful approximations to numbers, functions, or curves. The Leibniz approximation is a general framework for finding such approximations, that does not make direct use of the modern machinery of calculus. The idea is well-illustrated by the problem of finding approximations to the square root. This process was essentially known to the Babylonians.

Suppose we wish to compute $x = \sqrt{9.1}$ without the aid of a calculator. We correctly guess that $x \approx 3$. To obtain a better approximation, first note that x is supposed to satisfy the equation $x^2 = 9.1$. We obtain an improved approximation $x = 3 + dx$ by adding a small *correction* dx to the original guess $x = 3$. We now substitute this into the equation $x^2 = 9.1$, givin

$$\begin{aligned}(3 + dx)^2 &= 9.1\\ 9 + 6\,dx + dx^2 &= 9.1\\ 6\,dx + dx^2 &= 0.1.\end{aligned}$$

Now, this equation is not actually any easier to solve than the original equation $x^2 = 9.1$. However, since the correction dx is small (it is significantly less than 1), dx^2 will be extremely small. That is, dx^2 is *negligible*. The *Leibniz approximation* is to throw away these negligible higher degree terms, and just keep the linear term in dx. So, instead of solving the quadratic equation $6\,dx + dx^2 = 0.1$, we can solve the *linear* equation $6\,dx = 0.1$, so $dx = 0.1/6 \approx 0.017$.

Now, our new and improved guess is $x = 3 + dx = 3 + 0.017 = 3.017$.

1. Use the Leibniz approximation to approximate $\sqrt{4.2}$.

Answer: We want to solve $x^2 = 4.2$. With the initial guess of $x \approx 4$, we add a small correction: $x = 2 + dx$, and we want

$$(2 + dx)^2 = 4.2.$$

Expanding out the left-hand side gives

$$4 + 4\,dx + dx^2 = 4.2$$

Discarding the quadratic term,

$$4\,dx = 0.2$$

so $dx = 0.05$. Thus the improved guess is $x \approx 2.05$.

Example: We shall consider the circle

$$x^2 + y^2 = 25$$

and find the equation of the tangent line at the point $(3, 4)$. Suppose we start at the point $(3, 4)$, and move the x value from 3 to $3 + dx$, where dx is some small increment. To stay on the curve, the y value must go from 4 to $4 + dy$, where dy is another small increment.

In order to stay on the curve, we should have

$$(3 + dx)^2 + (4 + dy)^2 = 25.$$

Simplifying,

$$\begin{aligned}(9 + 6dx + dx^2) + (16 + 8dy + dy^2) &= 25\\ 25 + 6dx + 8dy + dx^2 + dy^2 &= 25\\ 6dx + 8dy + dx^2 + dy^2 &= 0.\end{aligned}$$

Now, if dx and dy are both small, dx^2 and dy^2 are very much smaller. Therefore, if we want a linear approximation to the curve, we should simply neglect these quadratic terms (that is, set them to zero!) The equation we then have is

$$6dx + 8dy = 0.$$

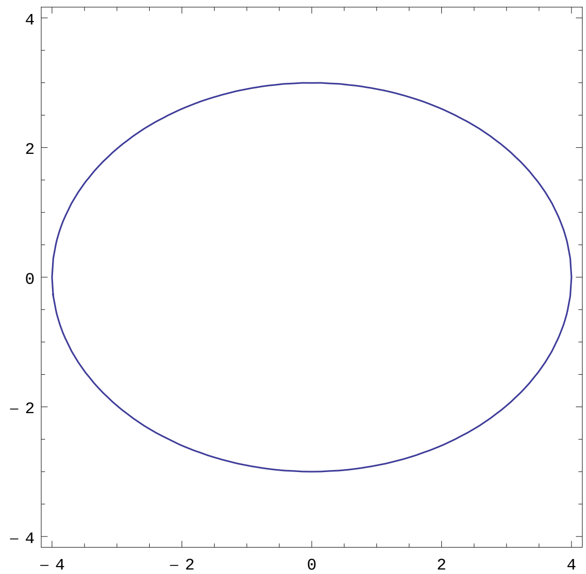
Since $dx = x - 3$ is the displacement from 3 and $dy = y - 4$, so the equation is

$$6(x - 3) + 8(y - 4) = 0.$$

This is the equation of a line, the *tangent line* to the circle at the point.

2. Consider the ellipse with equation

$$\frac{x^2}{16} + \frac{y^2}{9} = 1.$$



Notice that the point $(2.4, 2.4)$ lies on this ellipse. We will describe a way to find the tangent line to the ellipse at that point.

(a) Verify algebraically that $(2.4, 2.4)$ is a point on this ellipse. (This is not completely obvious!)

(b) Starting at the point $(2.4, 2.4)$, we increase the x value from 2.4 a small amount dx. In order to stay close to the curve, we should also increase y by a small amount dy. These quantities, dx and dy, should be related in such a way that the new point $(2.4 + dx, 2.4 + dy)$ continues to satisfy the equation of the ellipse:

$$\frac{(2.4 + dx)^2}{16} + \frac{(2.4 + dy)^2}{9} = 1.$$

On the above picture, mark the point $(2.4, 2.4)$, and illustrate a point $(2.4 + dx, 2.4 + dy)$ that satisfies this condition (where dx and dy are small). Also draw the secant line that connects the two points you have marked. Notice that if dx is positive, then dy is negative.

(c) Let's approach this algebraically. Starting with the equation

$$\frac{(2.4 + dx)^2}{16} + \frac{(2.4 + dy)^2}{9} = 1,$$

expand the two polynomials on the left-hand side of this equation, and only keep the first-order terms in dx and dy. (That is, throw away dx^2 and dy^2 as "negligible".)
Answer: $\frac{4.8dx}{16} + \frac{4.8dy}{9} = 0$.

(d) If you have done the algebra correctly above, then you should have an equation of the form $A\,dx + B\,dy = 0$, where A and B are two numbers. What are the values of A and B?

$A = 4.8/16 (= 0.3)$ and $B = 4.8/9 = 1.6/3 \approx 0.53$

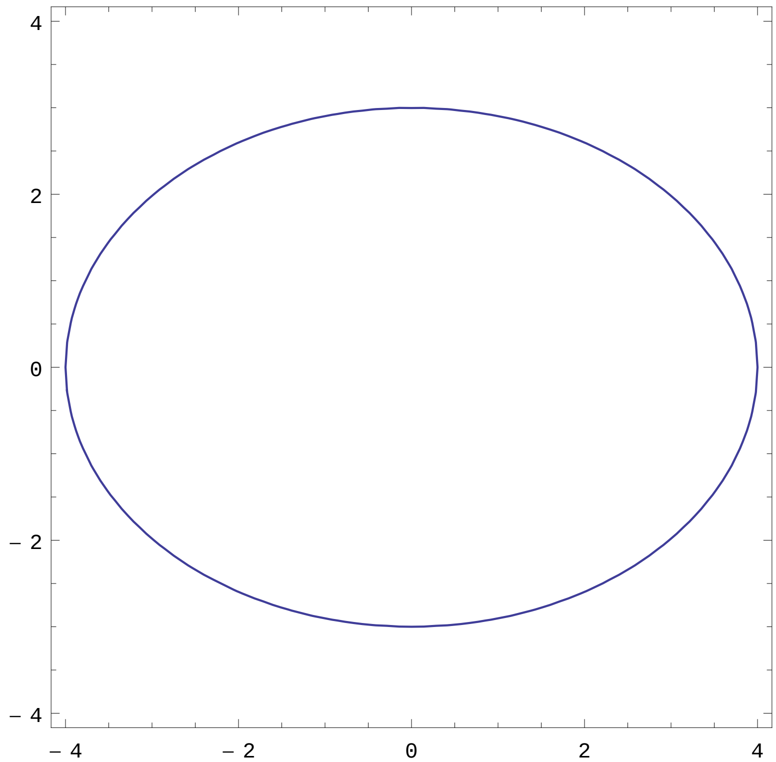
(e) Substitute $dx = (x - 2.4)$ and $dy = (y - 2.4)$ to obtain the equation of the tangent line.

Answer: $\frac{4.8(x-2.4)}{16} + \frac{4.8(y-2.4)}{9} = 0$.

(f) Next, solve the equation $A\,dx + B\,dy = 0$ for dy in terms of dx. The quantity $dy/dx = -A/B$ is the *slope of the tangent line* through the point $(2.4, 2.4)$. Write the equation of the tangent line through the point $(2.4, 2.4)$ and sketch it in the graph.

Answer: The tangent line is

$$y = 2.4 - \frac{9}{16}(x - 2.4)$$



3. Repeat this process to find the equation of the tangent line to the ellipse $4x^2 + y^2 = 25$ at the point $(2, 3)$.

Answer: Putting $x = 2 + dx$ and $y = 3 + dy$, we insist that the point (x, y) continue to be on the ellipse, so that

$$4(2 + dx)^2 + (3 + dy)^2 = 25$$

or

$$16 + 16\,dx + 4dx^2 + 9 + 6\,dy + dy^2 = 25$$

Cancelling the 25 and setting the quadratic terms to zero, this becomes

$$16\,dx + 6\,dy = 0.$$

Since, $x = x - 2$ and $dy = y - 3$, the equation of the tangent line is therefore:

$$16\,(x - 2) + 6\,(y - 3) = 0.$$

Historical note: The Leibniz approximation involves throwing away certain negligible terms. Gottfried Willhelm Leibniz felt that this could be justified by regarding differential quantities like dx and dy as "infinitesimal". His methods were met with a great deal of skepticism. Perhaps you share in that skepticism now. For a long time, though, the Leibniz approximation remained the way available to justify the results of calculus. It was not until the 19th century, 150 years after Leibniz's death, that the idea of a *limit* was introduced as a way of finally putting calculus on solid mathematical foundations.