Lecture 12: The derivative of a function

Jonathan Holland

Rochester Institute of Technology*

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*These slides may incorporate material from Hughes-Hallet, et al, "Calculus", Wiley

Finding a linear approximation

- Find the linear approximation to $f(x) = x^5 + x$ near the point $(1, 2)$.
- Note $(x^5 + x - 2) = (x - 1)(x^4 + x^3 + x^2 + x + 2)$
- Localize! $x^5 + x - 2 \approx 6(x - 1)$, so $x^5 + x \approx 2 + 6(x - 1)$ for x near 1.

The linear approximation to a rational function

Theorem

Let $f(x)$ be a rational function, and $x = a$ a regular point. Then there exists a (unique) linear function $L(x)$ such that

$$f(x) \approx L(x)$$

for x near a.

- To find the linear approximation, we can localize the rational function $f(x) - f(a)$ to get a linear function of the form mx, so that $f(x) \approx f(a) + mx$
- The slope of this linear function, m, is the derivative of $f(x)$ at $x = a$. That is, $m = f'(a)$.

Finding the linear approximation

Let $f(x)$ be a function. The linear approximation $L(x)$ for x near a is a linear function such that $f(x) \approx L(x)$ for x near a.
What this means is that $f(a) = L(a)$ and

$$\lim_{x \to a} \frac{f(x) - f(a)}{L(x) - L(a)} = 1.$$

Write $L(x) = m(x - a) + c$.
Then $c = L(a) = f(a)$.
To determine m, we have

$$\lim_{x \to a} \frac{f(x) - f(a)}{m(x - a)} = 1$$

so

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Activity

Consider the function $f(x) = x^5 + x$.
This function is one-to-one, and so has an inverse function.
Its range is $(-\infty, \infty)$, by the intermediate value theorem.
How would we find $f^{-1}(2.1)$?
How would we solve $x^5 + x = 2.1$?
Use the linear approximation! For x near 2, $f(x) \approx 2 + 6(x - 1)$
Set $2 + 6(x - 1) = 2.1$ and solve: $x = 1 + 0.1/6 \approx 1.017$

Differentiability

Definition

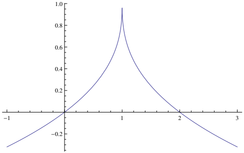
A function $f(x)$ is called *differentiable* at a point $x = a$ if there is a linear function $L(x)$ such that

$$\frac{f(x) - L(x)}{x - a} \to 0$$

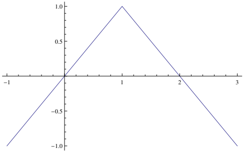
as $x \to a$.

- When the slope of $L(x)$ is not zero, this is equivalent to $f(x) \approx L(x)$ for x near a.
- Give examples of functions that are differentiable and non-differentiable. What this means in terms of the graph?

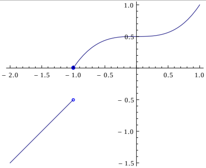
- A function is differentiable if when we zoom in far enough, the graph resembles a straight line
- Sometimes bumps in a curve do not smooth out, no matter how far in we zoom
- Corners, cusps, and discontinuities



A cusp



A corner



A discontinuity

Finding a tangent line: Newton–Cauchy style

Problem

Find the equation of the tangent line to the graph of the function $y = x^2 + x + 1$ *through the point* $(-2, 3)$.

Fact

The point-slope form of the tangent line to a graph $y = f(x)$ *through* $(a, f(a))$ *is* $y - f(a) = f'(a)(x - a)$.

- Here $a = -2$. We need to find $f'(-2)$.

$$f'(-2) = \lim_{\Delta x \to 0} \frac{f(-2+\Delta x) - f(-2)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(-2+\Delta x)^2 + (-2 +}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-4\Delta x + \Delta x^2 + \Delta x}{\Delta x} = \lim_{\Delta x \to 0} (-3 + \Delta x)$$

$$= -3$$

- The slope is -3, so the equation is $y - 3 = -3(x - (-2))$.

Example

Example

Let $f(x) = 1/x$ and find $f'(2)$.

- Before we do that, do we expect $f'(2)$ to be positive or negative?
- Calculate

$$\lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h} = \lim_{h\to 0} \frac{\left(\frac{2-(2+h)}{2(2+h)}\right)}{h}$$

$$= \lim_{h\to 0} \frac{-h}{2(2+h)h} = \lim_{h\to 0} \frac{-1}{2(2+h)} = \frac{-1}{4}$$

- Stay organized! Use parentheses!

The derivative as a function

Definition

Let $f(x)$ be a function. For each x in the domain of f, we calculate the limit:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

This defines a function, called the *derivative* of f.

- The quantity $f'(x)$ has units of slope. It is the slope of the tangent line to the graph at the point $(x, f(x))$.
- If $f(x) = ax + b$, then
  $$f'(x) = \lim_{h \to 0} \frac{(a(x+h) + b) - (ax + b)}{h} = \lim_{h \to 0} \frac{ah}{h} = a.$$
  (The tangent line has slope a at every point!)
- If $f(x) = x^2$, then $f'(x) = \lim_{h \to 0} \dfrac{(x+h)^2 - x}{h} = \cdots = 2x$.