Lecture 3: Review of trigonometry and inverse functions

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What is triangular and lives in a cave?
A triglodyte

* These slides may incorporate material from Hughes-Hallet, et al, “Calculus”, Wiley

Review of trigonometry

- Terms: **Radian**, **amplitude**, **period**, **sine**, **cosine**, **tangent**, **secant**
- Concepts. You should know...
  - Soh-Cah-Toa
  - On the unit circle, $\cos\theta$ is the x value and $\sin\theta$ is the y value
  - $\tan\theta$ is the slope of the line making an angle θ with the x axis
  - How to graph the sine, cosine, and tangent, including intercepts and (in the case of tangent) vertical asymptotes
- Skills:
  - Convert degrees to and from radians
  - Identify the period and aplitude of a function from an equation, description, or graph
  - Design a periodic function with a specific amplitude and period

Radians

- The formulas of calculus are simpler in radians.

Definition

A radian is defined to be the angle at the center of a unit circle which cuts off an arc of length 1, measured counterclockwise.

- $360^\circ = 2\pi$radians
- Application: computing arc lengths of sectors. What is the arc length of a sector of a circle of radius r, measuring θ radians?
- What is the area of a pizza slice, if the angle at the vertex is $\pi/6$ radians, and the diameter of the pizza is 24"?

Sine and Cosine

- The two basic trig functions are the sine and cosine, defined via the unit circle.
- For a point (x, y) that is θ radians on the unit circle measured in a counterclockwise sense from the x-axis, the cosine and sine are defined by $\cos\theta = x$ and $\sin\theta = y$. (https://www.geogebra.org/m/dngh2vcg)
- Since the unit circle is defined by $x^2 + y^2 = 1$, we have $\cos^2\theta + \sin^2\theta = 1$.
- As θ increases and P moves around the circle, the values of $\sin\theta$ and $\cos\theta$ oscillate between 1 and -1, and eventually repeat as P cycles back to a point it has already visited.
- If θ is negative, the angle is measures clockwise around the circle.

Amplitude, period, phase

- Graphs of sine and cosine: oscillate between a minimum of -1 and maximum of $+1$. (Amplitude=1)
- Periodic functions, period 2π.
- What are the x-intercepts? Note, sine and cosine are horizontal translations (called *phase shifts*) of each other.

Definition

For any periodic function of time, the

- *Amplitude* is half the distance between the maximum and minimum values (if it exists)
- *Period* is the smallest time needed for the funciton to execute one complete cycle.

Sinusoidal functions

- A function of the form $f(t) = A\sin(Bt)$ or $g(t) = A\cos(Bt)$ has amplitude $|A|$ and period $2\pi/|B|$.
- A function of the form $f(t) = A\sin(Bt) + C$ or $g(t) = A\cos(Bt) + C$ has a graph shifted vertically by C, and oscillates around this value.
- Examples: $y = 5\sin(2t)$, $y = -5\sin(t/2)$, $y = 1 + 2\sin t$.
- Find possible functions from graphs.

Modeling example

- High tide in Boston at midnight. Water level was 9.9 ft.
- Low tide was at about noon. The water level was 0.1 ft.
- Find a formula for the water level.
- More precisely, the difference between high and low tide is 12 hours, 24 minutes. Give a more accurate model.

Activity 1

Suppose the curve $y = \cos(2x)$ depicts a wave at time $t = 0$, much as if you could take a photograph that shows the oscillation in air pressure that is a sound wave. Graph this curve, then answer the following questions.

1. If length is measured in millimeters, and the number 2 has units of $\frac{\text{radians}}{\text{mm}}$, what is the wavelength of this wave?
2. Suppose the wave is traveling to the right at 340,290 mm per second. Answer the following questions with an equation, $y = \cdots$
    1. What curve depicts the wave after $t = 10^{-6}$ seconds?
    2. What curve depicts the wave after $t = 2 \times 10^{-6}$ seconds?
    3. What curve depicts the wave after $t = 3 \times 10^{-6}$ seconds?
    4. What curve depicts the wave after t seconds?

Activity 2

In this exercise you'll use the graph of a cosine function to mathematically describe a wave. Suppose x and y are measured in meters.

1. Suppose we take a photograph of the wave and find that it has a wavelength is 0.002 centimeters, and an amplitude of 0.07 cm. Determine numbers A and k so that the curve $y = A\cos(k\,x)$ depicts the wave.
2. Suppose the wave is traveling toward a detector at 100 meters per second. How many wave crests arrive at the detector per second?
3. What's the period of the wave as experienced by the detector?

Tangent function

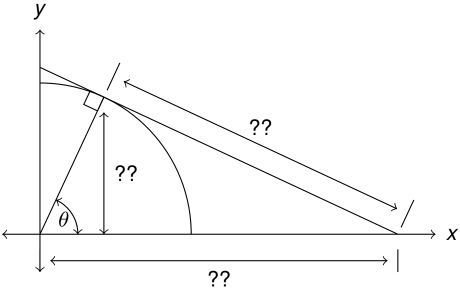
- If t is any number with $\cos t \neq 0$, then the tangent of t is defined by
$$\tan t = \frac{\sin t}{\cos t}.$$
- This is the slope of the line from the origin to the point $(\cos t, \sin t)$ of the unit circle.
- So, the tangent function $\tan \theta$ gives the slope of the line making an angle θ with the x axis!
- The tangent function is undefined at the t-intercepts of the cosine, where $\cos t = 0$, so at $t = \pm\pi/2, \pm 3\pi/2, \ldots$. It has vertical asymptotes at these points.
- The tangent function is periodic. What is the period?
- Does it make sense to talk about the amplitude of the tangent function?

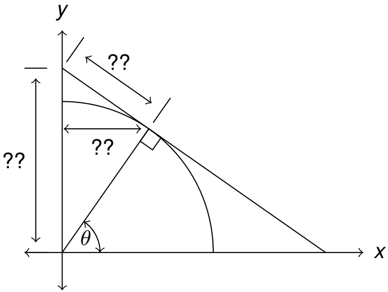
Soh-Cah-Toa, and all that

- A useful mnemonic for remembering how sine, cosine, tangent are related to the sides of a right triangle
- $$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}, \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$
  $$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$
- Other trig functions are defined by
  $$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}$$
  $$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$
- Note: Everything can be written in terms of sine and cosine. So these are the most fundamental.

Relation to the unit circle

Label each of the lengths on the following diagrams. The circle has a radius of 1 unit.





Inverse functions: preview

- Terms: **inverse function**, **one-to-one function**, **horizontal line test**, **arcsine**, **arctangent**, **natural logarithm**
- Skills:
  - Recognize when a function is one-to-one from an equation or graph;
  - Find an inverse function algebraically, from a table of values, or from a graph;
  - Use the arcsine to solve an equation for an unknown angle;
  - Use the natural logarithm, and the rules for natural logarithms, to solve an equation where the unknown variable appears in the exponent.

Inverse functions

- Bekele's running time, 2005 10k world record $t = f(d)$

d (meters)	$t = f(d)$ (seconds)
0	0.00
2000	315.63
4000	629.98
6000	944.66
8000	1264.63
10000	1577.53

- We could ask for distance as function of time $d = f^{-1}(t)$. For example, if we ask how far Bekele ran in the first 629.98 seconds, the answer is 4000 m, so $f^{-1}(629.98) = 4000m$.
- Going backwards in this way from numbers of seconds to numbers of meters gives f^{-1}, the *inverse function* of f. So $f^{-1}(t)$ is the number of meters Bekele ran during the first t

Which functions have inverses?

- A function $y = f(x)$ is called *one-to-one* if every output value y comes from one and only one input value x.
- Graphically, this leads to the *horizontal line test*: a function is one-to-one if every horizontal line intersects the graph in at most a single point.

Definition

If the function f is one-to-one, then its inverse is defined as follows:

$$f^{-1}(y) = x \qquad \text{means} \qquad y = f(x).$$

Graphs of inverse functions

- The function $f(x) = x^3$ is invertible. Why?
- To find the inverse, solve $y = x^3$ for x: $x = y^{1/3}$.
- So $f^{-1}(y) = y^{1/3}$. Or, if we want to call the independent variable x, $f^{-1}(x) = x^{1/3}$.
- The graphs $y = x^3$ and $y = x^{1/3}$ are reflections of each other about the line $y = x$.

Fact

The graph of the inverse function $y = f^{-1}(x)$ comes from the graph of the function $y = f(x)$ by switching the x and y coordinates of every point.

Formulas for inverse functions

Example

Let $f(x) = \frac{x+1}{x-2}$. Find $f^{-1}(x)$.

Solution

- *Start with* $y = \frac{x+1}{x-2}$*. The inverse function comes by exchanging the* x *and* y *variables, so for the inverse function we have* $x = \frac{y+1}{y-2}$*.*
- *Now solve this equation for* y*.*
- $x(y-2) = y+1 \implies xy - 2x = y + 1 \times xy - y = 2x + 1 \implies y = \frac{2x+1}{x-1}$
- *So* $f^{-1}(x) = \frac{2x+1}{x-1}$

Domain and range

Theorem

If f is a one-to-one function, then the domain of f^{-1} is the range fo f and the range of f^{-1} is the domain of f>

Example

Find the range of $f(x) = \frac{x+1}{x-2}$.

Solution

We have $f^{-1}(x) = \frac{2x+1}{x-1}$. The domain of f^{-1} is $(-\infty, 1) \cup (1, \infty)$, which is the same as the range of f.

The number e

- The most frequently used base for an exponential function is the number $e \approx 2.71828\ldots$.
- This turns out to be the most convenient base for many applications. e^x appears on most scientific calculators.
- Any exponential growth function can be written

$$P = P_0 e^{kt}$$

where $k > 0$. Any exponential decay function can be written

$$P = P_0 e^{-kt}$$

- k is called the *continuous rate* of growth (or decay)
- The amount P grows during a small time dt is $dP \approx kP\,dt$

Compound interest

- Suppose we compound an investment n times throughout the year, with a nominal rate of r.
- Then the *effective* annual rate is
$$e_n(r) = -1 + \left(1 + \frac{r}{n}\right)^n.$$
- Suppose we invest at a nominal rate of 3% ($r = 0.03$), then effective rate, compounding monthly times is
$$e_{12}(0.03) = -1 + \left(1 + \frac{0.03}{12}\right)^{12} \approx 0.0304$$
- So the effective rate is slightly higher, 3.04%.
- The functions $e_n(r)$ are a family of polynomials in r.
- Note graphically that $e_n(r) \leq -1 + e^r$ for all n and r.
- $\lim_{n\to\infty} e_n(r) = -1 + e^r$

The natural logarithm

- The natural logarithm of x, written $\ln x$, is the power of e needed to get x. That is, $\ln x = c$ means $e^c = x$.

Fact

- *To convert from $P = P_0 a^t$ to $P = P_0 e^{kt}$, set $k = \ln a$.*
- *To convert from $P = P_0 e^{kt}$ to $P = P_0 a^t$, set $a = e^k$.*

Properties of natural logarithms

- $\ln(ab) = \ln a + \ln b$
- $\ln(a/b) = \ln a - \ln b$
- $\ln(a^p) = p \ln a$
- $\ln e^x = x$
- $e^{\ln x} = x$
- $\ln 1 = 0$

Example

If $2^{x^2} = 4 \cdot 3^x$, solve for x.

Solution

Note: By the order of operations, $2^{x^2} = 2^{(x^2)} \neq (2^x)^2$ *and* $4 \cdot 3^x = (4)(3^x) \neq 12^x$. *Taking natural logarithms,*

$$\begin{aligned} \ln(2^{x^2}) &= \ln(4 \cdot 3^x) \\ x^2 \ln(2) &= \ln(4) + \ln(3^x) \\ x^2 \ln(2) &= \ln(4) + x \ln(3) \\ \ln(2)x^2 - \ln(3)x - \ln(4) &= 0 \end{aligned}$$

So, by the quadratic formula, $x = \frac{\ln(3) \pm \sqrt{(\ln 3)^2 + 4 \ln(2) \ln(4)}}{2 \ln(2)}$

Solving trigonometric equations

- Sometimes we need to find the angle corresponding to a given sine. (Draw a picture.)
- For example, find θ such that $\sin\theta = 0$. Or θ such that $\sin\theta = 0.3$.
- Answers to second $\theta \approx 0.305, 2.84, 0.305 \pm 2\pi, 2.84 \pm 2\pi, \ldots$
- For each equation we pick out the solution between $-\pi/2$ and $\pi/2$ as the preferred solution.
- Preferred solution of $\sin\theta = 0$? $\sin\theta = 0.3$?

Arcsine

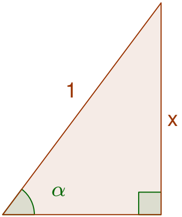
Definition

For $-1 \leq y \leq 1$, $\arcsin y = x$ means $\sin x = y$ with $-\pi/2 \leq x \leq \pi/2$.

x	$\sin x$
$-\frac{\pi}{2}$	$-1.$
-1	-0.841471
$-\frac{1}{2}$	-0.479426
0	$0.$
$\frac{1}{2}$	0.479426
1	0.841471
$\frac{\pi}{2}$	$1.$

x	$\sin^{-1} x$
-1	$-\frac{\pi}{[illegible]}$

Visualizing the arcsine



- Draw a right triangle with one angle $\sin^{-1} x$. Indicate the lengths of the various sides.
- Compute $\tan(\sin^{-1} x)$ using this triangle.

Arctangent

- The inverse tangent is the inverse function for the piece of the tangent function having the domain $-\pi/2 < x < \pi/2$.

Definition

For any y, $\arctan y = x$ means that $\tan x = y$ with $-\pi/2 < x < \pi/2$.

- Graphs. Note horizontal asymptotes.
- An "application": Experience levels in Halo series
- Geometrical interpretation.