Handout: Antiderivatives and indefinite integrals

This handout concerns the concept of an *antiderivative*. If $f(x)$ is a given function, then an antiderivative of $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$.

1. Find an antiderivative of $f(x) = x^2$.

2. Show that $-\cos(x)$ and $1 - \cos(x)$ are both antiderivatives of $\sin(x)$. What is the most general antiderivative of $\sin(x)$?

3. Differentiate to show that $\ln(1 + x)$ is an antiderivative of $^1/_{(1+x)}$, but $\ln(1 + x^2)$ is *not* an antiderivative of $^1/_{(1+x^2)}$.

4. Suppose $f''(x) = 2x - 4$, $f'(1) = 3$ and $f(2) = 8$. Find a formula for $f(x)$.

The next set of exercises concerns the indefinite integral. The indefinite integral is the most general antiderivative of a function. For example, the indefinite integral of the function $f(x) = x$ is the function $F(x) = \frac{x^2}{2} + C$ where C is a free constant. We write the indefinite integral using the notation

$$\int f(x)\,dx$$

For example,

$$\int x\,dx = \frac{x^2}{2} + C$$

5. Evaluate the indefinite integral $\displaystyle\int x^2\,dx$.

The rules for antidifferentiation are more difficult than the rules for differentiation. In fact, not every function has a rule going the other way, so there are functions that we will not be able to integrate exactly.

The basic rules, however, can be obtained by reading the rules for differentiation in reverse. Here is the table of the basic rules for integration:

Differential rule	Integral rule
$d(x^{n+1}) = (n+1)x^n dx$	$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \ (n \neq -1)$
$d(\ln\|x\|) = \frac{dx}{x}$	$\int \frac{dx}{x} = \ln\|x\| + C$
$d(af(x)) = adf(x)$	$\int af(x)\, dx = a \int f(x)\, dx$
$d(f(x) + g(x))$	$\int (f(x) + g(x))\, dx = \int f(x)\, dx + \int g(x)\, dx$
$d(e^x) = e^x dx$	$\int e^x \, dx = e^x + C$
$d(\sin x) = \cos x \, dx$	$\int \cos x \, dx = \sin x + C$
$d(\cos x) = -\sin x \, dx$	$\int \sin x \, dx = -\cos x + C$
$d(\arctan x) = \frac{dx}{1+x^2}$	$\int \frac{dx}{x^2+1} = \arctan x + C$
$d(\arcsin x) = \frac{dx}{\sqrt{1-x^2}}$	$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$

6. Use the rules to find the indefinite integral $\displaystyle\int \left(x^3 - 7x^2 + \frac{1}{x} \right) dx$

- **There is no product rule for integration.** That is there is no rule that will allow us to calculate the integral $\int f(x)g(x)\, dx$ from the integrals of f and g.

7. Give an example of a pair of functions $f(x)$ and $g(x)$ such that $\int f(x)g(x)\, dx \neq \left(\int f(x)\, dx\right)\left(\int g(x)\, dx\right)$

- **There is no quotient rule for integration.** There is no rule that will allow us to calculus the integral $\int f(x)/g(x)\, dx$ from the integrals of f and g.

There are many other "non-rules" for integration: things that we wished were true, but aren't. We will later learn about a rule called the substitution rule which can sometimes help to evaluate certain integrals. But *unlike derivatives*, there is no algorithm for calculating integrals, because our list of rules does not tell us how to deal with all of the possible operations that go into the parse tree of an expression (for example, multiplication and division).

In fact, there are relatively simple functions that *cannot be integrated* using the rules developed in calculus. So, when encountering an integral "I don't know how to find the answer" is sometimes a legitimate answer: performing integration requires *non-deterministic* reasoning. If you're skilled with the rules (and "non-rules") of manipulation of algebraic expressions, then a principle that sometimes works is:

- **When in doubt, expand it out.** That is, expand all products and quotients in order to attempt the sum rule term-by-term.

8. Find the integral $\displaystyle\int (x+1)(x^2+x+1)\, dx$.