9. Limits of rational functions: a user's guide

This worksheet is intended as a practical guide to calculating limits. We shall just be concerned with rational functions, functions of the form

$$f(x) = \frac{p(x)}{q(x)}.$$

A key fact is:

- If $x = c$ is in the domain of the rational function $f(x) = \frac{p(x)}{q(x)}$, then

$$\lim_{x \to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}$$

1. Compute

$$\lim_{x \to 2} \frac{x^2 + 3x + 4}{x^2 - 9}.$$

This key fact is not always good enough, though, because sometimes the point c is not in the domain of the rational function. Here is what to do in that case:

- If $x = c$ is not in the domain of the rational function $f(x) = \frac{p(x)}{q(x)}$, then factor the numerator and denominator, and cancel any common factors.

Once we have done this, it might be that we have produced a new rational function with the point $x = c$ "filled in" on the graph.

2. Compute the following limits.

   (a) $\lim_{x \to 2} \frac{x^2-4}{x-2}$

   (b) $\lim_{x \to 2} \frac{x^2-3x+2}{x^2-4}$

   Answers: (a) 4, (b) 1/4

   However, it might happen that, despite our best efforts, $x = c$ is not in the domain of the function that we get. The following theorem guarantees that (assuming we have factored everything properly), this implies that $x = c$ is a vertical asymptote.

   Theorem 1. *If $q(x)$ is a polynomial such that $q(c) = 0$, then $q(x)$ must have a factor of $x - c$.*

   So, for calculating the limit $\lim_{x \to c} \frac{p(x)}{q(x)}$ there are only four possibilities:

   - $p(c)$ and $q(c)$ are both nonzero, in which case $\lim_{x \to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}$
   - $p(c) = 0$ and $q(c)$ is not zero, in which case $\lim_{x \to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)} = 0$ (a *zero*)
   - $p(c) = q(c) = 0$, in which case $p(x)$ and $q(x)$ *have a factor of $x - c$ in common*
   - $p(c) \neq 0$ and $q(c) = 0$, in which case $p(x)/q(x)$ has a vertical asymptote (a *pole*)

   In the first two cases, we can evaluate the limit by direct substitution. In the third case, we need to cancel all common factors of $x - c$. This just leaves the last case.

We want to be able to give a sense of how the function approaches the asymptote $x = c$. From each side of the asymptote, there is a limit.

- We will write
$$\lim_{x\to c^-} f(x) = +\infty$$
if $f(x)$ becomes arbitrarily large and positive as x approaches c *from the left* (this is what the c^- refers to).
- We will write
$$\lim_{x\to c^-} f(x) = -\infty$$
if $f(x)$ becomes arbitrarily large and negative as x approaches c *from the left.*
- We will write
$$\lim_{x\to c^+} f(x) = +\infty$$
if $f(x)$ becomes arbitrarily large and positive as x approaches c *from the right* (this is what the c^+ refers to).
- We will write
$$\lim_{x\to c^+} f(x) = -\infty$$
if $f(x)$ becomes arbitrarily large and negative as x approaches c *from the right.*

3. Draw an example of each of these four possibilities.
4. Find
$$\lim_{x\to 0^-} \frac{-1}{x^3} = +\infty$$
and
$$\lim_{x\to 0^+} \frac{-1}{x^3} = -\infty$$
5. Find
$$\lim_{x\to 2^-} \frac{-1}{(x-2)^3} = +\infty$$
and
$$\lim_{x\to 2^+} \frac{-1}{(x-2)^3} = -\infty$$
(Hint: Think about how the graph of $\frac{-1}{(x-2)^3}$ is related to the graph of $\frac{-1}{x^3}$.)
6. Find
$$\lim_{x\to -2^-} \frac{-1}{(x+2)^2} = -\infty$$
and
$$\lim_{x\to -2^+} \frac{-1}{(x+2)^2} = -\infty$$

For a more general rational function, you can find the limits of a rational function at an asymptote by first determining the behavior at the asymptote. Here, you can use what we practiced on an earlier workshop. For example, to find
$$\lim_{x\to -2^+} \frac{x+3}{(x+1)(x+2)^2}$$

localize at $x=-2$ by plugging $x=-2$ into all of the non-zero terms in the factorization. So near $x=-2$, we have the approximation

$$\frac{x+3}{(x+1)(x+2)^2} \approx \frac{(-2+3)}{(-2+1)(x+2)^2} = \frac{-1}{(x+2)^2}$$

and so, by the previous problem,

$$\lim_{x\to -2^+} \frac{x+3}{(x+1)(x+2)^2} = -\infty.$$

7. Find

$$\lim_{x\to 1^-} \frac{x-2}{(x-1)^3}$$

and

$$\lim_{x\to 1^+} \frac{x-2}{(x-1)^3}$$

Answers: Near $x=1$, we have $\frac{x-2}{(x-1)^3} \approx \frac{1-2}{(x-1)^3} = -\frac{1}{(x-1)^3}$, so

$$\lim_{x\to 1^-} \frac{x-2}{(x-1)^3} = +\infty$$

$$\lim_{x\to 1^+} \frac{x-2}{(x-1)^3} = -\infty$$

A final kind of limit is *a limit at infinity.* The behavior of a rational function for large values of x is determined by the leading power of x in the numerator and denominator. For example, to calculate

$$\lim_{x\to\infty} \frac{2x^2+x+1}{3x^2+2x},$$

it is sufficient to look at the x^2 terms in the numerator and denominator:

$$\lim_{x\to\infty} \frac{2x^2+x+1}{3x^2+2x} = \lim_{x\to\infty} \frac{2x^2}{3x^2} = \frac{2}{3}.$$

8. Find

$$\lim_{x\to\infty} \frac{3x^2+3x+1}{x^2-1}$$

Answer: $3/1 = 3$

If the degree of the denominator is greater than the degree of the numerator, then the limit of a rational function is zero.

9. Find

$$\lim_{x\to\infty} \frac{1}{x}$$

Answer: 0

10. Find

$$\lim_{x\to\infty} \frac{3x^3+3x+1}{x^5+2x^3-1}$$

Answer: 0

11. There are four different types of limits: one-sided, two-sided, one-sided infinite, limit at infinity. Give an example of each.

$\lim_{x\to 1^+} \frac{x-1}{|x-1|} = 1$, $\lim_{x\to 2} \frac{x-2}{x^2-4} = 1/4$, $\lim_{x\to 0^+} \frac{1}{x} = +\infty$, $\lim_{x\to\infty} \frac{1}{x} = 0$.