Limits and continuity

1. In each part below, determine formulas for $f(x)$ and $g(x)$ so that the statement is satisfied.

   (a) $\lim_{x\to 0}(fg)(x)$ exists but $\lim_{x\to 0} f(x)$ does not.

   (b) $\lim_{x\to 0}\left(\frac{f}{g}\right)(x)$ exists but $\lim_{x\to 0} g(x) = 0$.

   Examples: (a) $f(x) = \sin(1/x), g(x) = x$; (b) $f(x) = x^2$, $g(x) = x$.

2. Draw the curves $y = \cos(x)$ and $y = 2 - \cos(x)$. Then draw the graph of a function f so that $\cos(x) \le f(x) \le 2 - \cos(x)$. Based on your graph, determine the value of $\lim_{x\to 2\pi} f(x)$.

   Ans: The limit is 1.

3. Suppose f is the function defined below. Determine $\lim_{x\to 0} f(x)$. *(Hint: graph $y = x^2$ and $y = 3x$.)*

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is irrational} \\ 3x & \text{otherwise} \end{cases}$$

   Ans: 0

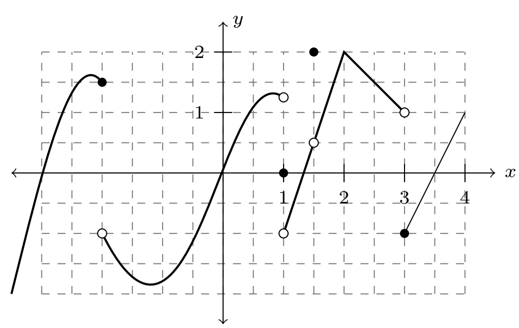
4. This exercise refers to the graph below.

   (a) Is there any value of x where the one-sided limits are the same but the function is not continuous?

   (b) We say that a function is *left-continuous* at $x = c$ if $f(c) = \lim_{x\to c^-} f(x)$. Find any/all points at which f is discontinuous but is left-continuous.

   (c) We say that a function is *right-continuous* at $x = c$ if $f(c) = \lim_{x\to c^+} f(x)$. Find any/all points at which f is discontinuous but is left-continuous.

   (d) We say that f has a jump discontinuity at $x = c$ if $\lim_{x\to c^+} f(x)$ and $\lim_{x\to c^-} f(x)$ both exist but are different. Find any/all points where f has a jump discontinuity.



   (a) Yes, at $x = 1.5$ (b) At $x = -2$ (c) $x = 3$ (d) $x = -2$, $x = 1$, and $x = 3$

5. In each case, determine a value for $f(c)$ that makes the function continuous at $x = c$

   (a) $f(x) = \frac{x^2+1x-6}{x^2-x-2}$, $c = 2$

   (b) $f(x) = \frac{x^2+4x+7}{x^2+x+12}$, $c = 0$

   (c) $f(x) = \frac{x^3-1}{\sqrt{x}-1}$, $c = 1$

   (d) $f(x) = \frac{\cos(4x)-1}{\cos(10x)-1}$, $c = 0$

   Ans: (a) 5/3, (b) 7/12 (the function is already continuous), (c) 6 (requires the difference of two cubes!)

6. Suppose f and g are as seen below.

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{otherwise} \end{cases} \qquad g(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find $\lim_{x\to 0} f(x)$ or explain why it does not exist.

   (b) Find $\lim_{x\to 0} g(x)$ or explain why it does not exist.

(c) Identify any/all points at which f is continuous.

(d) Identify any/all points at which g is continuous.

7. Design a function f that is discontinuous at $x = 2$ but is continuous everywhere else.

8. A *fixed point* is a number c at which $f(c) = c$. Suppose $f(x) = 1 + \cos(x) - x^2$. Without using your calculator, and without finding c, show that f has a fixed point.

   Solution. Let $g(x) = f(x) - x$. A fixed point of f is a zero of g. Now, g is continuous. Also $g(0) = 2 > 0$ and $g(-\pi) = -\pi - \pi^2 < 0$. So somewhere between $x = -\pi$ and $x = 0$ there is a zero of $g(x)$ by the intermediate value theorem, and so a fixed point of $f(x)$.

9. Prove that you were exactly 3 feet tall at some point in your life. What assumptions do you need to make about your height as a function of time?

   Ans: When I was born, I was less than three feet tall, and now I am more than three feet tall. I was exactly three feet tall, by the intermediate value theorem, assuming that my height is a continuous function of my age.

10. Suppose f is continuous on the interval $I_1 = [a, b]$, with $f(a) < 0$ and $f(b) > 0$. The Intermediate Value Theorem guarantees that f has a root somewhere in $[a, b]$. Without further information, our best guess is that the root is at the midpoint of the interval, $x_1 = {}^{(a+b)}/_2$. In the *bisection algorithm* for finding roots, we use this "best guess" to gather more information about f. After calculating $f(x_1)$, we know that a root is in the subinterval across which we see a change in the sign of $f(x)$, either $[a,\, x_1]$ or $[x_1,\, b]$. Define I_2 to be that subinterval, and repeat the process: calculate the midpoint of I_2, which we'll label as x_2, and based on $f(x_2)$ determine which half of I_2 has a root of f. Call that subinterval $I_3 \ldots$. The process continues until we either find a root (which is unlikely, generally speaking) or we reach a stopping criterion.

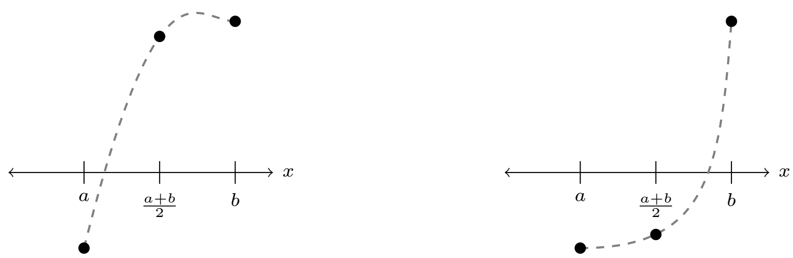


Figure: *Diagrams depicting possible scenarios. A third scenario is possible. What is it?*

(a) Demonstrate your understanding of the bisection method by finding x_3 when $f(x) = 2x - 12$ and $I_1 = [4, 7]$.

$$x_1 = 11/2, x_2 = 25/4, x_3 = 47/8$$

(b) Suppose $f(x_1) > 0$, so we select the left half of $[a, b]$ to be I_1. Is it possible for f to have a root in the other subinterval? If so, draw a diagram showing how this might happen. If not, explain why not.

Ans: Yes it is possible. The function may dip below the x axis between a and x_1.

(c) Let's denote by x_* the actual root of f in $[a, b]$. A reasonable stopping criterion is that x_n is "close enough" to x_*. We quantify "close enough" numerically with a positive number δ (which is typically very small). Write an inequality that says the distance between x_* and x_k does not exceed δ.

$|x_k - x_*| < \delta$

(d) Although we don't know the distance between x_k and x_* exactly we know that it's no larger than the radius of I_k. Suppose $\delta = 0.01$. Show that the number of steps required to get "close enough" to x_* is always the same, regardless of f. How many steps is that?

$\lfloor \log_2[(b-a)/\delta] \rfloor$