# Lecture 21: Maxima and minima

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*These slides may incorporate material from Hughes-Hallet, et al, “Calculus”, Wiley

# Preview

- Terms: **stationary point**, **critical point**, **local maximum/minimum**, **absolute maximum/minimum**, **extremum**, **inflection point**
- Skills: There are three kinds of problem, each can be solved by making ***two different kinds of table***.
  - To find the **local** extrema of a function $y = f(x)$, first find the stationary points by solving $f'(x) = 0$ for $x$, and then make a table of signs of $f'(x)$ in each of those intervals. Use this table to decide whether the stationary points are maxima/minima/neither. (Note: This uses a table constructed *from values of the derivative* $f'(x)$.)
  - To find the **absolute** extrema of a function $y = f(x)$ on a compact interval $[a, b]$, make a table of values *of the function* $f$ sampled at the endpoints of the interval $a$ and $b$, as well as any stationary points in the interior of the interval. (Note: This uses a table constructed *from the values of the function* $f(x)$.)
  - To find the inflection points of a function $y = f(x)$, first solve $f''(x) = 0$ for $x$, and then make a table of signs of $f''(x)$ in each of those intervals. (Note: This uses a table constructed *from values of the second derivative* $f''(x)$.)

# What derivatives tell us about a function and its graph

- If $f' > 0$ on an interval, then $f$ is increasing on that interval.
- If $f' < 0$ on an interval, then $f$ is decreasing on that interval.
- When we graph a function on a calculator, we may miss some important features.
- How do we decide on an appropriate interval to graph?
- Information from the derivative can help to identify regions with interesting behavior.

*Example*

Consider the function $f(x) = x^3 - 9x^2 - 48x + 52$.

# $f(x) = x^3 - 9x^2 - 48x + 52$

- What are the stationary points of $f$?
- $f'(x) = 3x^2 - 18x - 48 = 3(x^2 - 6x - 16) = 3(x+2)(x-8)$
- Make a table...

| | $x < -2$ | $-2 < x < 8$ | $8 < x$ |
|---|---|---|---|
| Sign of $f'$ | $+$ | $-$ | $+$ |
| Behavior of $f$ | Increasing | Decreasing | Increasing |

# *Local maxima and minima*

*Definition*

Suppose $p$ is a point in the domain of $f$:

- $f$ has a *local minimum* at $p$ if $f(p)$ is less than or equal to the values of $f$ for all points near $p$.
- $f$ has a *local maximum* at $p$ if $f(p)$ is greater than or equal to the values of $f$ for all points near $p$.

# *How do we detect a local maximum or minimum?*

- In the preceding example, $f(x) = x^3 - 9x^2 - 48x + 52$.
- We found the stationary points $x = -2$ and $x = 8$.
- These played a key role in leading us to the local maxima and minima.

*Theorem*

*Suppose $f$ is defined on an interval and has a local maximum or minimum at the point $x = a$, which is not an endpoint of the interval. If $f$ is differentiable at $x = a$, then $f'(a) = 0$. Thus $a$ is a stationary point.*

# *A warning...*

- Not every stationary point is a local maximum or minimum.
- An example is $f(x) = x^3$
- What is the stationary point?
- Why isn't this a local maximum? minimum?

# Testing for local maxima and minima at a stationary point

*Theorem*

*Suppose $p$ is a stationary point of a differentiable function $f$.*

- *If $f'$ changes from negative to positive at $p$, then $f$ has a local minimum at $p$.*
- *If $f'$ changes from positive to negative at $p$, then $f$ has a local maximum.*

# Examples

- $f(x) = \dfrac{1}{x(x-1)}$
- Why are there no local maxima or minima of $g(x) = \sin x + 2e^x$ for $x \geq 0$? What about $x \leq 0$?

# Higher derivatives

- $f''(x)$ is the derivative of $f'(x)$, called the *second derivative* of $f$.
- Example: If $s(t) = \textit{position}$, $s'(t) = ?$, $s''(t) = ?$
- $f'''(x)$ is the third derivative, etc.
- $s'''(t)$ is called the *jerk*: it represents a sudden change in acceleation that one feels as a “jerk”. Slamming on the brakes, e.g.
- Snap, crackle, pop!
- Leibniz notation: $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, etc

# What does the second derivative tell us?

- If $f'' > 0$ on an interval, then $f'$ is *increasing* over that interval.
- Draw a picture.
- If $f'' > 0$ on an interval, then the graph of $f$ is *concave up* on the interval.
- If $f'' < 0$ on an interval, then $f'$ is *decreasing* over that interval.
- If $f'' < 0$ on an interval, then the graph of $f$ is *concave down* on the interval.

# Examples

- $f(x) = x^3$
- $f(x) = x^4$
- $f(x) = x^3 - 9x^2 - 48x + 52$
- $f(x) = \frac{1}{x(x-1)}$

# Inflection points

*Definition*

An *inflection point* is a point where the concavity of a graph changes. (So from up to down or down to up.)

*Example*

- $f(x) = x^3$ has an inflection point at 0
- Where are the inflection points of $f(x) = x^3 - 9x^2 - 48x + 52$?

# Absolute maxima and minima

*Definition*

Let $f$ be a function. The point $x = p$ is an *absolute* maximum of $f$ if $f(x) \leq f(p)$ for all values of $x$. The point $x = q$ is an *absolute* maximum of $f$ if $f(x) \leq f(q)$ for all values of $x$.

Examples:

- Let $f(x) = x^2$ on $(-\infty, \infty)$. What is the absolute minimum? maximum? (if any)
- Is the local minimum of $f(x) = x^2 - 9x^2 - 48x + 52$ an absolute minimum? Is the local maximum an absolute maximum?

# *Local versus absolute*

- A local extremum does not need to be an absolute extremum.
- A function may fail to have absolute extrema.
- Draw the graph of a differentiable function on the interval $(-1, 1)$ that has a single stationary point at $x = 0$, which is a local minimum, but no absolute maximum.
- Is there a function on $[-1, 1]$ that has no absolute maximum or minimum?

# Extreme value theorem

*Theorem*

*Let $f$ be a continuous function on the closed and bounded interval $[a, b]$. Then $f$ has an absolute maximum and an absolute minimum in that interval. Furthermore, if $f$ is differentiable in the interior of the interval, then the absolute extrema must occur either at the endpoints $a, b$, or at some stationary point(s) in $(a, b)$.*

# Absolute extrema: the table method

*Example*

Find the absolute extrema of the function $f(x) = x^2 - 2x$ on the interval $[-1, 4]$.

*Solution*

*We compute $f'(x) = 2x - 2 = 0$ when $x = 1$ (stationary pt).*

*Now make a table:*

| $x$ | $f(x)$ |
|---|---|
| -1 | 3 |
| 1 | -1 |
| 4 | 8 |

*So the absolute minimum is at $x = 1$, and has the value $f(1) = -1$. The absolute max is at $x = 4$, with the value $f(4) = 8$.*