MATH-181 Example Term Projects

Version: January 20, 2018

Contents

Introduction

This document provides examples of ***projects*** for Project-Based Calculus I. These differ from worksheets and modules in their scale, deliverables, and guidance. Projects typically present students with a problem but offer minimal guidance about its solution. Due to this fact, students often need to invest substantial out-of-class effort, and projects always require a formal written report in which students delineate both the problem and its solution.

Acknowledgements

The projects in this document were developed, written, and coded by Carl Lutzer and H. Tim Goodwill. The idea for The Tube was contributed by Paul Wenger.

License and Agreement

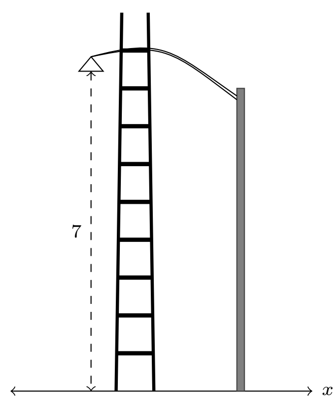
Project: The Lamp of Unluckiness

Suppose a lamp is suspended at a height of 7 meters. It's a rather dark afternoon in autumn, so the lamp is on when a worker climbs up to check the lamp's photovoltaic cells.

1. When the worker pulls out a pen to make a note, he drops it. Miraculously, the pen remains horizontal as it falls straight down from an altitude of 7 meters, directly below the lamp. If the pen is 15 cm long, determine the rate at which the length of its shadow is changing when it's at an altitude of 6 meters. Report your answer in meters per second. (You may treat the lamp as a point-source of light. The initial velocity of the pen is 0, and you can safely treat the acceleration due to gravity as being a constant $a = -9.8\,\frac{\text{m}}{\text{sec}}$ per second.)

2. After dropping his pen, our worker remembers that he has a pencil in his jacket pocket.☺ The tip of the pencil is broken☹, so he drops it. Suppose the pencil is 10 cm long, maintains a horizontal orientation as if falls, and is falls straight downward, one meter to the left of the lamp. If the pencil was dropped from a height of 6.5 meters, determine the rate at which the length of its shadow is changing when it's at an altitude of 5 meters. (Again, take the initial velocity to be 0, and $a = -9.8\,\frac{\text{m}}{\text{sec}}$ per second.)

3. Now bereft of writing instruments, the worker decides that it's a good time for a pick-me-up snack. So he balances himself on the ladder and withdraws a snack-bag that contains an orange and several donut holes.☺ But apparently the snack-bag was not properly closed, and as he withdraws it from his pocket, the contents start spilling out.☹ Our unlucky hero fails to catch them. Rather, he bobbles the falling fruit, and ends up giving it a vertical knock that sends the orange upward at 1 meter per second. The orange travels straight up, and then drops straight down toward the ground, passing right between his hands. If the vertical knock happened at a height of 6 meters, 2 meters right of the lamp, determine the rate at which the length of the orange's shadow is changing when it's at a height of 4 meters.



▷ Deliverables

The deliverable for this project is a report in which you use the concepts and techniques of calculus to answer question #3 (questions #1 and #2 are warm-up exercises).

- Your report should be expository in nature, with clear and concise, well-written sentences.

- Mathematical ideas and techniques from this class should be demonstrated explicitly, not simply referenced.
- A student from another section of this class should be able to read your report and understand your results without asking you questions.
- If you use information from any source other than what is provided here, you must cite the source and provide a reference in a bibliography.
- If you use MATLAB or any other software, well-commented code should appear as an appendix.
- The first section of your report should be titled Declaration of Independent Work and should contain the following comment:

  > *The solution we present in this report is our own. It represents our own creative, independent problem solving efforts, including our implementation of the mathematical (and computational) methods.*

  Before handing in your report, all group members must sign this statement in pen.

Project: The Tube

In order to increase tourism, the city of Chicago (USA) has decided to dig a tunnel through the earth to Rome (Italy), both of which are at 41.9° N latitude. The tunnel will be coated with the famous frictionless varnish that covers tabletops in physics classrooms throughout the world, so that when people jump or fall into the tunnel earth's gravity will pull them through.

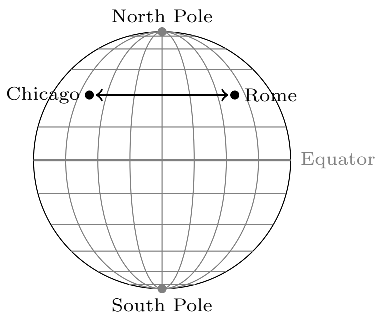


Figure 1: The earth viewed from a point far above the equator.

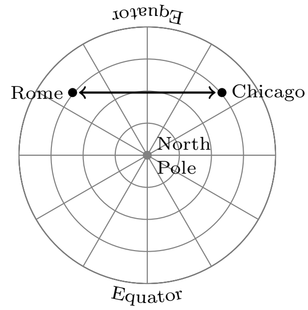
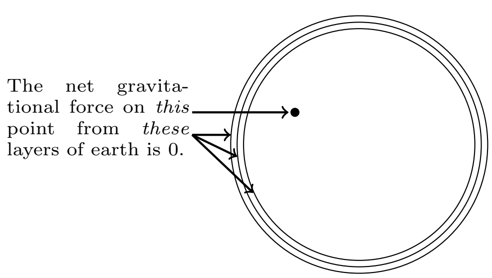


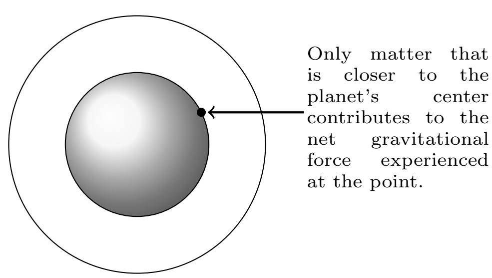
Figure 2: The earth viewed from a point far above the North Pole.

In this project, your job is to determine the duration of a trip from Chicago to Rome, and the traveler's maximum velocity, so that your company knows what kind of amenities it can sell to customers (goggles, flame-retardant suits, emotional support animals, goggles and flame-retardant suits for the emotional support animals,...).

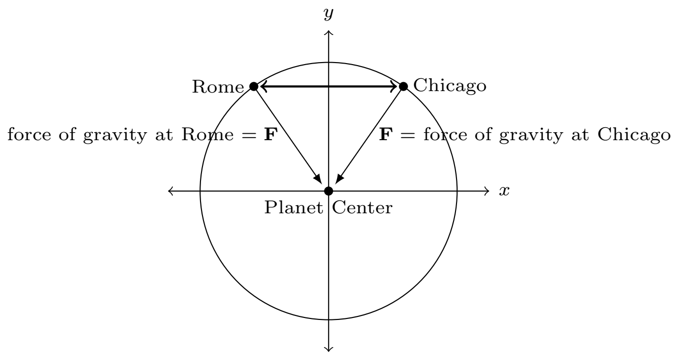
▷ Helpful Facts and Admissible Assumptions

- You may assume that Earth's surface is spherical, and that the planet has a uniform density.
- The web site `https://nssdc.gsfc.nasa.gov/planetary/factsheet/earthfact.html` has useful data about the Earth.
- When you are in Earth's interior, only earth that is closer to the planet's center (i.e., closer than you are) contributes to the force of gravity that you experience. The net contribution from the spherical shells of planet-stuff above you is 0. (Mathematical proof of this fact relies on something called the Divergence Theorem, which you learn when studying the calculus of vector fields.) It will be useful for you to develop a formula for the force due to gravity at a distance of r meters from the planet's center (where r is less than the radius of the planet).

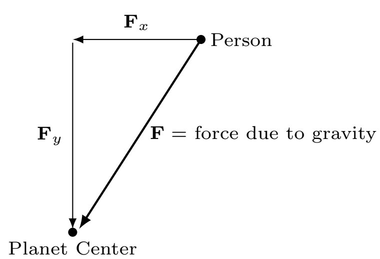




- Figure 1 and Figure 2 are standard ways of showing a globe, but we can make the mathematics of this scenario much easier by working in the plane that that passes through Chicago, Rome, and the planet's center. This plane intersects the (spherical) globe in a great circle.



- Newton's Second Law states that an object's acceleration is related to the net force it experiences by the equation $F_{\text{net}} = ma$ where m is the object's mass. In this case, the only force acting on a traveler is gravity, which we can write as the sum of a "horizontal" and a "vertical" force: $\mathbf{F} = \mathbf{F}_x + \mathbf{F}_y$ (see diagram below, and compare to diagram above). The vertical aspect of the force plays no role because the tube offers no friction, so only the horizontal component of gravity contributes to accelerating the object.



- The volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.
- Chicago is at 41.94° N latitude, 12.5° E longitude.
- Rome is at 41.94° N latitude, 87.62° W longitude.

▷ Deliverables

The deliverable for this project is a typed report in which you answer the question of how long a trip takes, and the maximum velocity of a traveler.

- Your report should be expository in nature. It should consist of clear and concise, well-written sentences that delineate the reasoning and calculations that lead to your answers.
- Mathematical ideas and techniques should be demonstrated explicitly, not simply referenced. Equations should be treated as grammatical elements, not display pieces.

- A student from another section of this class should be able to read your report and understand your results without asking you questions.
- If you use information from any source other than what is provided here, you must cite the source and provide a reference in a bibliography.
- If you use MATLAB or any other software, well-commented code should appear as an appendix.
- The first section of your report should be titled Declaration of Independent Work and should contain the following comment:

  > *The solution we present in this report is our own. It represents our own creative, independent problem solving efforts, including our implementation of the mathematical (and computational) statement.*

  Before handing in your report, all group members must sign this statement in pen.

▷ Instructor's Notes: The Tube

Thanks to Paul Wenger for communicating the basic idea of this project.

1. When a person of mass m kg is in the tube, at a distance of r from the center of the planet, the planetary mass that contributes to the gravitational force is

$$M = \frac{4}{3}\pi r^3 \rho\,,$$

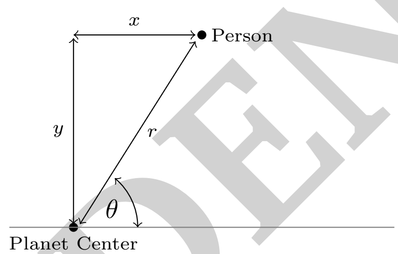
where $\rho = 5514\,\frac{\text{kg}}{\text{m}^3}$ is the average density of the planet, so the gravitational force acting on the person is

$$F_g = -\frac{GMm}{r^2} = -\frac{Gm}{r^2} \times \left(\frac{4}{3}\pi r^3 \rho\right) = -\frac{4\pi G m \rho}{3}\, r\,. \tag{1}$$

The horizontal component of this force is

$$F_x = F_g \cos(\theta) \tag{2}$$

where θ is the angle shown in the diagram below.



In the diagram we see that $\cos(\theta) = \frac{x}{r}$. Combining this with equations (1) and (2) allows to express the horizontal force as

$$F_x = -\frac{4\pi G m \rho}{3}\, r \times \frac{x}{r} = -\frac{4\pi G m \rho}{3}\, x\,. \tag{3}$$

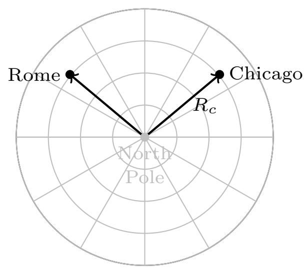
2. Newton's Second Law tells us that

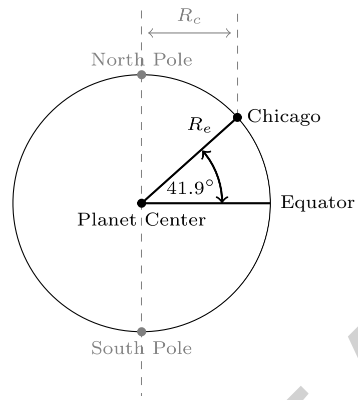
$$\begin{aligned} ma &= F_x \\ \ddot{x} &= -\frac{4\pi G \rho}{3}\, x \end{aligned} \tag{4}$$

Students should recognize equation (4) as describing a sine, cosine, or linear combination of them. Given the initial condition $x(0) = x_0$ (see below), we see that

$$x(t) = x_0 \cos\left(\sqrt{\frac{4\pi G\rho}{3}}\, t\right)\,.$$

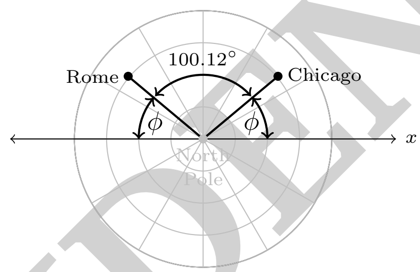
3. Because Chicago is at a latitude of 41.9°, its orthogonal distance to Earth's north-south axis is $R_c = R_e \cos(41.9°)$, where $R_e = 6{,}378{,}137$ meters is the equatorial radius of the Earth.





The cities are separated by $12.5° + 87.62° = 100.12°$ longitude. This leaves $79.88°$ in a half-circle, which we depict as two arcs of $\phi = 39.94°$ in the diagram below. From this information we can determine

$$x_0 = R_c \cos(39.94°) = R_e \cos(41.9°) \cos(39.94°) \approx 0.5706764264 \times R_e = 3{,}639{,}852\,\text{m}.$$



Project: Dealing with Uncertainty

You've already learned about minimizing mathematical "energy" to find the best line $y = mx$ through data in which there is only uncertainty in y, but because solar radiation and output from the photovoltaic array are both measured quantities, there is likely to be at some uncertainty in both numbers. This requires us to adapt our technique; we redefine the term *residual* to mean the perpendicular distance between a point of data and the line $y = mx$ (see Figure 3). After that, we set

$$E = (r_1)^2 + (r_2)^2 + (r_3)^2 + \cdots + (r_n)^2$$

and find the value of m that minimizes E.

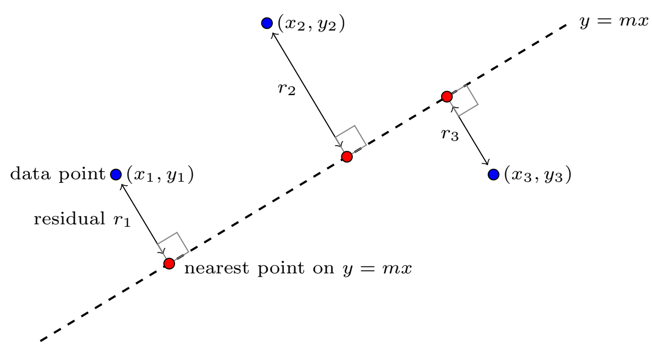


Figure 3: A residual is the perpendicular distance between observed data and the line of fit.

In this project, your job is to **use the concepts and techniques of calculus** to develop the details of an energy-minimization method for finding the best line, $y = mx$, through data in which there is uncertainty in both the x and y coordinates. Then you should apply your method to the data in `GIS-Solar.mat` and compare the resulting line to the line from the MATLAB workshop (plot both lines with the data, and discuss their relative merits).

▷ Notes

- The coordinates of the points on $y = mx$ that are nearest to the observed data (respectively) will depend on m.
- Formulas should be derived from first principles, using elementary analytic geometry (rather than citing sources on the internet).

▷ Deliverables

The deliverable for this project is a report in which you demonstrate how the concepts and techniques of calculus allow us to find a line of best fit.

- Your report should be expository in nature, with clear and concise, well-written sentences.
- Mathematical ideas and techniques from this class should be demonstrated explicitly, not simply referenced.

- A student from another section of this class should be able to read your report and understand your results without asking you questions.
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