

Appendix A – Introduction: Uncertainties, Error Propagation, and Graphing

1. Systematic and Random Errors.

No measurement is ever exact. The **accuracy** (correctness) and **precision** (number of significant figures) of a measurement are always limited by the degree of refinement of the apparatus used, the measurement technique, the skill of the observer, and the nature of the quantity being measured. In doing an experiment, we are trying to establish the best value for a certain quantity or trying to validate a theory. This requires that we establish a **range of possible true values** based on our limited number of measurements.

Why should repeated measurements of a single quantity give different values? Mistakes on the part of the experimenter are possible, but we do not include these in our discussion. A careful researcher should not make mistakes! (Or at least she or he should recognize them and correct the mistakes.)

We use the synonymous terms **uncertainty**, **error**, or **deviation** to represent the variation in measured data. Two types of errors are possible. **Systematic error** is the result of a poorly calibrated device, or a measuring technique which always makes the measured value larger (or smaller) than the “true” value. An example would be using a steel ruler at liquid nitrogen temperature ($-196^{\circ}\text{C} = -320^{\circ}\text{F}$) to measure the length of a rod. The ruler will contract at low temperatures and therefore overestimate the true length. Careful design of an experiment will allow us to eliminate or correct for systematic errors.

Even when systematic errors are eliminated there will remain a second type of variation in measured values of a single quantity. These remaining deviations will be classed as **random errors** and can be dealt with in a statistical manner. This document does not teach statistics in any formal sense, but it should help you to develop a working methodology for treating errors.

2. Relative and Absolute Errors.

Every measured quantity has an inherent uncertainty (or error) associated with it. For example, if D represents the measured diameter of a cylinder, then the associated uncertainty in the diameter is written as δD . The quantity δD is called the **absolute error** while $\delta D/D$ is called the **relative** or **fractional uncertainty**. **Percentage error** is the fractional error multiplied by 100%. In practice, either the percentage error or the absolute error may be provided. Thus in machining an engine part the tolerance is usually given as an absolute error, while electronic components are usually given with a percentage tolerance.

3. Determining Random Errors.

How can we estimate the uncertainty of a measured quantity? Several approaches can be used, depending on the application.

a) Instrument Limit of Error (ILE) and Least Count

The **least count** is the smallest division that is marked on the instrument. Thus a meter stick will have a least count of 1.0 mm, a digital stop watch might have a least count of 0.01 s.

The **instrument limit of error (ILE)** is the precision to which a measuring device can be read, and is always equal to or smaller than the least count. Very good measuring tools are calibrated against standards maintained by the National Institute of Standards and Technology.

The Instrument Limit of Error is generally taken to be the least count or some fraction ($1/2$, $1/5$, $1/10$) of the least count). You may wonder which to choose, the least count or half the least count, or something else. No hard and fast rules are possible, instead you must be guided by common sense. If the space between the scale divisions is large, you may be comfortable in estimating to $1/5$ or $1/10$ of the least count. If the scale divisions are closer together, you may only be able to estimate to the nearest $1/2$ of the least count, and if the scale divisions are very close you may only be able to estimate to the least count.

b) Estimated Uncertainty

Often other uncertainties are larger than the ILE. We may try to balance a simple beam balance with masses that have an ILE of 0.01 grams, but find that we can vary the mass on one pan by as much as 3 grams without seeing a change in the indicator. We would use half of this as the **estimated uncertainty**, thus getting an uncertainty of ± 1.5 g.

Another good example is determining the focal length of a lens by measuring the distance from the lens to the screen. The ILE may be 0.1 cm, however the depth of field may be such that the image remains in focus while we move the screen by 1.6 cm. In this case the estimated uncertainty would be half the range or ± 0.8 cm.

c) Average Deviation: Estimated Uncertainty by Repeated Measurements

The statistical method for finding a value with its uncertainty is to repeat the measurement several times, calculate the **average** and find either the **average deviation** or the **standard deviation**.

Suppose we repeat a measurement several times and record the different values. We can then find the average value, here denoted by t_{av} , and use it as our best estimate of the reading. How can we determine the uncertainty? Let us use the data shown in Table 1 as an example. Column 1 shows a set of time measurements in seconds.

Table 1. Values showing the determination of average and average deviation in a measurement of time. Notice that to get a non-zero average deviation we must take the absolute value of the deviation.		
Time t (s)	Deviation $(t - t_{av})$ (s)	Absolute value of deviation $ t - t_{av} $ (s)
7.4	-0.2	0.2
8.1	0.5	0.5
7.9	0.3	0.3
7.0	-0.6	0.6
$t_{av} = 7.6$	$\{t - t_{av}\}_{av} = 0.0$	$\{ t - t_{av} \}_{av} = 0.4$

A simple average of the times is the sum of all values ($7.4 + 8.1 + 7.9 + 7.0$) divided by the number of readings (4), which yields 7.6 sec.

Column 2 of Table 1 shows the deviation of each time from the average, $(t - t_{av})$. A simple average of these is zero, and does not give any new information.

To get a non-zero estimate of deviation we take the average of the **absolute values** of the deviations, as shown in Column 3 of Table 1. We will call this the **average deviation**.

For a second example, consider a measurement of length shown in Table 2. The average and average deviation are shown at the bottom of the table. Round the uncertainty to one significant figure and then round the average so that it is precise to the same power of 10 as the uncertainty. Thus the average length with average deviation is (15.5 ± 0.1) m. The length is now written in “**proper form**” (discussed in more detail later in this Introduction).

Table 2. Example of finding an average length and an average deviation in length. The values in the last row have an excess of significant figures . The result is reported as (15.5 ± 0.1) m.	
Length x (m)	$ x - x_{av} $ (m)
15.4	0.06667
15.2	0.26667
15.6	0.13333
15.7	0.23333
15.5	0.03333
15.4	0.06667
Average 15.46667 m	± 0.133333 m

d) Conflicts in the above.

In some cases we will get an ILE, an estimated uncertainty, and an average deviation and we will find different values for each of these. We will be pessimistic and **take the largest of the three values as our uncertainty**. For example we might measure a mass required to produce standing waves in a string with an ILE of 0.01 grams and an estimated uncertainty of 2 grams. We use 2 grams as our uncertainty.

The proper way to write the answer (“proper form”) is

1. Choose the largest of (i) ILE, (ii) estimated uncertainty, and (iii) average deviation.
2. Round the uncertainty to 1 significant figure.
3. Round the answer so that it is precise to the same power of 10 as the uncertainty.
4. Put the answer and its uncertainty in parentheses, and put the power of 10 and unit **outside** the parentheses. **ALWAYS INCLUDE THE UNITS!**

4. What is the range of possible values?

When you see a number reported as (7.6 ± 0.4) s, your first thought might be that all the readings lie between 7.2 s ($= 7.6 - 0.4$) and 8.0 sec ($= 7.6 + 0.4$). A quick look at the data in Table 1 shows that this is not the case: only 2 of the 4 readings are in

this range. Statistically we expect 68% of the values to lie in the range of $x_{av} \pm \delta x$, and 95% to lie within $x_{av} \pm 2 \delta x$. In the first example, all the data lie between 6.8 (= 7.6 – 2*0.4) and 8.4 (= 7.6 + 2*0.4) sec. In the second example, 5 of the 6 values lie within two deviations of the average. **As a rule of thumb for this course we usually expect the actual value of a measurement to lie within two deviations from the mean.**

You can learn much more about confidence levels by taking a statistics course.

How do we use the uncertainty? Suppose you measure the density of calcite as $(2.65 \pm 0.04) \text{ g/cm}^3$. The textbook value is 2.71 g/cm^3 . Do the two values agree? Since the text value is within the range of two deviations from the average value you measure, then you claim that your value agrees with the text. If you had measured the density to be $(2.65 \pm 0.01) \text{ g/cm}^3$, then you would be forced to admit your value disagrees with the text value.

5. Propagation of Errors.

Suppose two measured quantities x and y have uncertainties, δx and δy , determined by procedures described in previous sections: we would report $(x \pm \delta x)$ and $(y \pm \delta y)$. A new quantity, z , is calculated from the measured quantities x and y . What is the uncertainty, δz , in z ? For the purposes of this course we will use a simplified version of a proper statistical treatment.

The uncertainty in a calculated value is the difference between the largest possible value and the best value (average value). The largest possible value of the calculated quantity is determined by using the largest or smallest of the entering values to yield the largest calculated value. Let's consider a few examples.

For example, suppose the diameter D and height H of a cylinder are measured and you are asked to determine the volume V and its uncertainty. The measured quantities are:

$$D = (25.3 \pm 0.1) \text{ mm}$$

$$H = (55.0 \pm 0.5) \text{ mm}$$

The volume of the cylinder is given by

$$V = \frac{1}{4} \pi D^2 H = \frac{1}{4} \pi (25.3 \text{ mm})^2 (55.0 \text{ mm}) = 27,649.90 \text{ mm}^3$$

The maximum value of the volume is calculated using the largest possible values for the diameter and height. Thus,

$$V_{\max} = \frac{1}{4} \pi (D + \delta D)^2 (H + \delta H) = \frac{1}{4} \pi (25.4 \text{ mm})^2 (55.5 \text{ mm}) = 28,122.27 \text{ mm}^3$$

Therefore, the uncertainty in the volume is

$$\delta V = V_{\max} - V = 28,122.27 \text{ mm}^3 - 27,649.90 \text{ mm}^3 = 472.37 \text{ mm}^3$$

Now, round the uncertainty in the volume to one significant figure ($5 \times 10^2 \text{ mm}^3$) and then round the volume so that it is precise to the same power of 10 as the uncertainty ($276 \times 10^2 \text{ mm}^3$). Finally, write the volume in "proper form" (value and uncertainty inside the parenthesis, with same power of ten and units outside the parenthesis):

$$V = (2.76 \pm 0.05) \times 10^4 \text{ mm}^3$$

As another example, suppose the time T to fall a distance y are measured and you are asked to calculate the acceleration a . The measured quantities are

$$y = (1.83 \pm 0.02) \text{ m}$$

$$T = (0.61 \pm 0.01) \text{ s}$$

The acceleration is calculated using

$$a = \frac{2y}{T^2} = \frac{2(1.83 \text{ m})}{(0.61 \text{ s})^2} = 9.8361 \frac{\text{m}}{\text{s}^2}$$

The maximum acceleration is obtained by using the largest value of the distance (making the **NUMERATOR LARGE** makes the calculated result large) and the smallest value of the time (making the **DENOMINATOR SMALL** makes the calculated result large). Thus, the maximum acceleration is

$$a_{\text{max}} = \frac{2(y + \delta y)}{(T - \delta T)^2} = \frac{2(1.85 \text{ m})}{(0.60 \text{ s})^2} = 10.2778 \frac{\text{m}}{\text{s}^2}$$

The uncertainty in the acceleration is therefore

$$\delta a = a_{\text{max}} - a = 10.2778 \frac{\text{m}}{\text{s}^2} - 9.8361 \frac{\text{m}}{\text{s}^2} = 0.4417 \frac{\text{m}}{\text{s}^2}$$

Round the uncertainty in the acceleration to one significant figure, then adjust the value of the acceleration so that it is precise to the same power of 10 as the uncertainty; finally, write the acceleration in “proper form”: $a = (9.8 \pm 0.4) \text{ m/s}^2$.

6. Rounding answers in regular and scientific notation (“proper form”).

In the above examples we were careful to round the answers to an appropriate number of significant figures. **The uncertainty must be rounded to one significant figure.** Then the answer should be rounded so that it is precise to the same power of 10 as the uncertainty.

When the answer is given in scientific notation, the uncertainty should be given in scientific notation with the **same power of ten**. Thus, if

$$z = 1.43 \times 10^6 \text{ s and } \delta z = 2 \times 10^4 \text{ s,}$$

the value is written as

$$z = (1.43 \pm 0.02) \times 10^6 \text{ s.}$$

This notation makes the range of values most easily understood. The following is not correct proper form and is hard to understand at a glance:

$$z = (1.43 \times 10^6 \pm 2 \times 10^4) \text{ s. Don't write like this!}$$

7. Significant Figures.

Use the rules for error propagation when measured quantities with uncertainties are available. If uncertainties are not available, then use standard rules for **significant figures**.

A significant figure is any digit from 1 to 9 and any zero which is not a place holder. Thus, in 1.3502 there are 5 significant figures since the zero is not serving as a place holder (it's between two non-zero numbers). There are 3 significant figures in the number 0.00322, the first three zeros are just place holders. However the number 1350 is ambiguous. You cannot tell if there are 3 significant figures with the 0 only used to hold the units place or if there are 4 significant figures and the zero in the units place was actually measured to be zero.

How do we resolve ambiguities that arise with zeros when we need to use zero as a place holder as well as a significant figure? Suppose we measure a length as 8000 cm. Written this way we cannot tell if there are 1, 2, 3, or 4 significant figures. **To make the number of significant figures apparent we use scientific notation.** For example, $8 \times 10^3 \text{ cm}$ has one significant figure and $8.00 \times 10^3 \text{ cm}$ has three significant figures.

We start then with numbers, each with their own number of significant figures and compute a new quantity. How many significant figures should be in the final answer? In doing running computations we maintain numbers to many figures, but we must report the FINAL answer only to the proper number of significant figures.

In the case of addition and subtraction, we can best explain with an example. Suppose one object is measured to have a mass of 9.9 g and a second object is measured on a different balance to have a mass of 0.3163 g. What is the total mass? We write the numbers with question marks at places where we lack information. Thus 9.9???? g and 0.3163? g. Adding them with the decimal points lined up we see

$$\begin{array}{r} 09.9???? \\ 00.3163? \\ \hline 10.2???? = 10.2 \text{ g.} \end{array}$$

The short rule for addition and subtraction is that the answer will contain the same number of decimal places as the entering number with the least number of decimal places (the least precise number). In the above example, 9.9 is known to the 1st decimal place (precision to the tenths place) and 0.3163 is known to the 4th decimal place, so the result is given to the 1st decimal place.

In the case of multiplication or division we can use the same idea of unknown digits. Thus the product of 3.413? and 2.3? can be written in long hand as

$$\begin{array}{r} 3.413? \\ 2.3? \\ \hline 6826? \\ 10239? \\ \hline 78????? = 7.8????? = 7.8 \end{array}$$

The short rule for multiplication and division is that the answer will contain a number of significant figures equal to the number of significant figures in the entering number having the least number of significant figures. In the above example 2.3 had 2 significant figures while 3.413 had 4, so the answer is given to 2 significant figures.

It is important to keep these concepts in mind as you use calculators with 8 or 10 digit displays if you are to avoid mistakes in your answers and to avoid the wrath of physics instructors everywhere. A good procedure to use is to use all digits (significant or not) throughout calculations and only round off the answer to the appropriate number of "sig figs".